Finite Elements for Lattice Field Theory on a Sphere

(Radial Lattice Quantization of Conformal Field Theory on the Lattice* ?)

$$\mathbb{R}^d \to \mathbb{R} \times \mathbb{S}^{d-1}$$

Richard Brower

Yale, June 20, 2014 QCDNA VII

*RCB, G. Fleming, H. Neuberger, M. Cheng & Andrew Gasbarro (last talk today)

Radial Quantization: *Early History*

- S. Fubini, A. Hanson and R. Jackiw PRD 7, 1732 (1972)

  Abstract: A field theory is quantized covariantly on Lorentz-invariant surfaces. Dilatations replace time translations as dynamical equations of motion. .... The Virasoro algebra of the dual resonance model is derived in a wide class of 2-dimensional Euclidean field theories.

- J. Cardy J. Math. Gen 18 757 (1985).

  Abstract: The relationship between the correlation length and critical exponents in finite width strips in two dimensions is generalised to cylindrical geometries of arbitrary dimensionality d. For d > 2 these correspond however, to curved spaces. The result is verified for the spherical model

Radial Quantization

Evolution: $$H = P_0 \text{ in } t \implies D \text{ in } \tau = \log(r)$$

$$ds^2 = dx^\mu dx_\mu = e^{2\tau}[d\tau^2 + d\Omega^2]$$

Can drop
Weyl factor!

$$\mathbb{R}^d \to \mathbb{R} \times \mathbb{S}^{d-1}$$

$$\text{"}time\text{"} \quad \tau = log(r) \;, \quad \text{"}mass\text{"} \quad \Delta = d/2 - 1 + \eta$$

$$D \to x_\mu \partial_\mu = r\partial_r = \frac{\partial}{\partial \tau}$$

Motivation

- (near) Conformal Field Theories, interesting for
  - BSM composite Higgs
  - AdS/CFT weak-strong duality
  - Model building & Critical Phenomena in general

- Lattices on $\mathbb{R}^d$ are problematic:
  - scales are exponentially divergent.

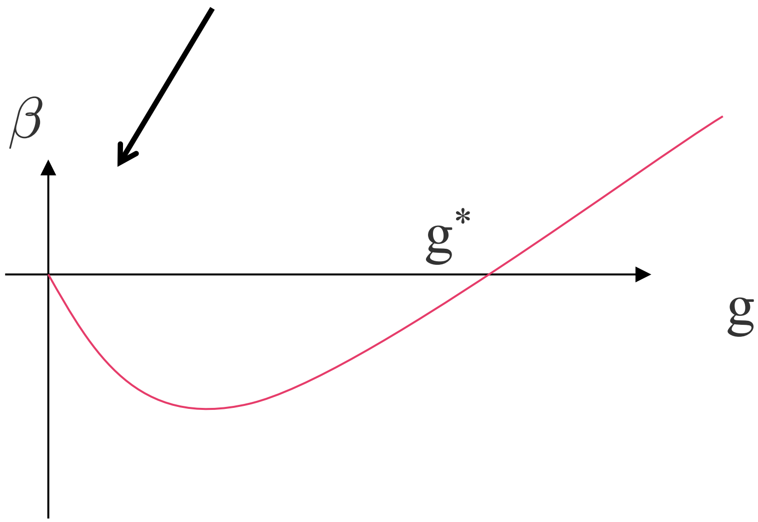
- Linear Hypercubic vs Exponential Radial Lattice

$$a < \Delta r < L \quad \text{vs} \quad a < \Delta log(r) < L$$

An IR fixed point can emerge already in the two-loop – function as you increase the number N_f of fermions.

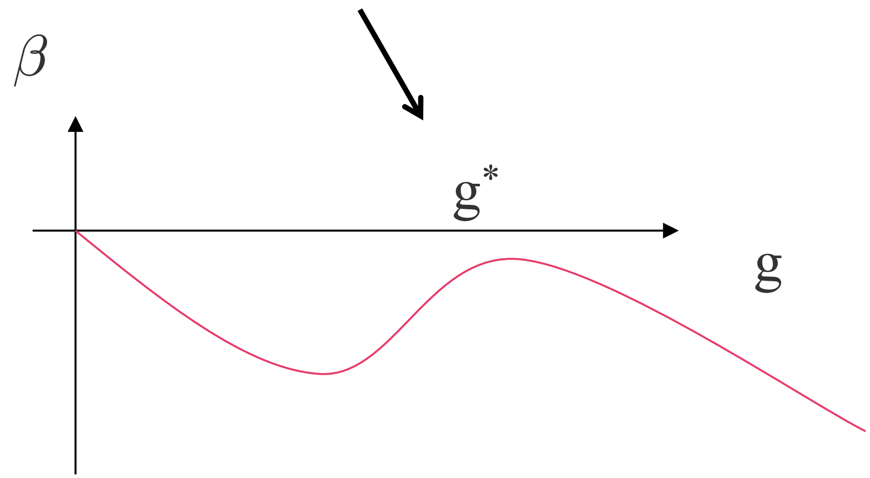
$$\beta(g) = b_0 g^3 + b_1 g^5 + \cdots$$

$b_0 < 0$ for $N_f < 11N_c/2 = 16.5$



Conformal Window:
16.5> N_f > N^*_f

$b_1 > 0$ for $N_f > 153/14$



Near conformal
N_f < N^*_f , close to N^*_f

Since 2007 Lattice Strong Dynamics (LSD) collaboration has begun to explore BSM electro-weak physics

Heechang Na
James Osborn

Rich Brower
Michael Cheng
Meifeng Lin
Claudio Rebbi
Oliver Witzel

Ethan Neil

Sergey Syritsyn

Michael Buchoff
Chris Schroeder
Pavlos Vranas

Ron Babich
Mike Clark

Joe Kiskis

David Schaich

Saul Cohen

Tom Appelquist
George Fleming
Gennady Voronov

Euclidean Conformal Field Theories

O(d+1,1) adds Dilations and Inversion to Poincare transformations

$$x_\mu \to \lambda x_\mu \quad , \quad x_\mu \to \frac{x_\mu}{x^2}$$

Algebra: $K_\mu : (inv \to trans \to inv)$

$$[K_\mu, \mathcal{O}(x)] = i(x^2\partial_\mu - 2x_\mu x^\nu \partial_\nu + 2x_\mu \Delta)\mathcal{O}(x)$$

$$[D, \mathcal{O}(x)] = i(x^\mu \partial_\mu - \Delta)\mathcal{O}(x)$$

$$[D, P_\mu] = -iP_\mu \quad , \quad [D, K_\mu] = +iK_\mu \quad , \quad [K_\mu, P_\mu] = 2iD$$

Exact CFT: Power Law Correlator

Conformal correlator: $\langle \phi(x_1)\phi(x_2)\rangle = C\frac{1}{|x_1 - x_2|^{2\Delta}}$

$$\begin{aligned} r_1^\Delta r_2^\Delta \langle \phi(\tau_1,\Omega_1)\phi(\tau_2,\Omega_2)\rangle &= C\frac{1}{[r_2/r_1 + r_1/r_2 - 2\cos(\theta_{12})]^\Delta} \\ &\simeq Ce^{-(\log(r_2) - \log(r_1)\Delta} \\ &= Ce^{-\tau\Delta} \end{aligned}$$

With $|x_1 - x_2|^2 = r_1 r_2[r_2/r_1 + r_1/r_2 - 2\cos(\theta_{12})]$

as $\tau = \log(r_2) - \log(r_1) \to \infty$

CFT are highly constrained

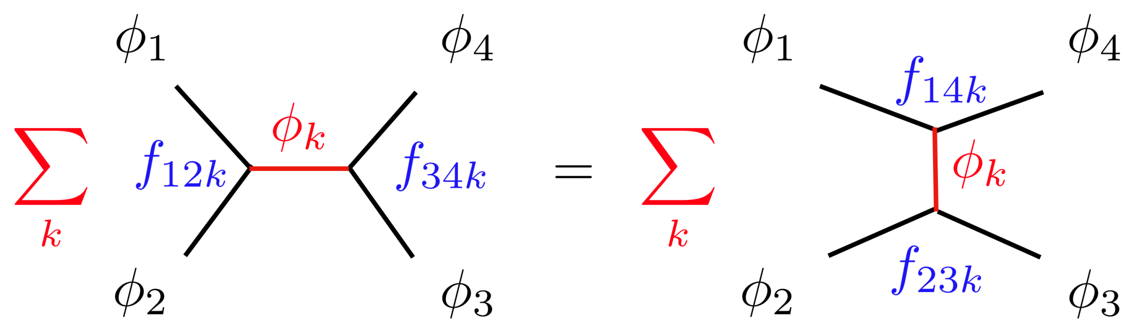
(i.e. Data: spectra + couplings to conformal blocks)

Exact 2 and 3 correlators

$$\langle \phi(x_1)\phi(x_2)\rangle = C\frac{1}{|x_1-x_2|^{2\Delta}}$$

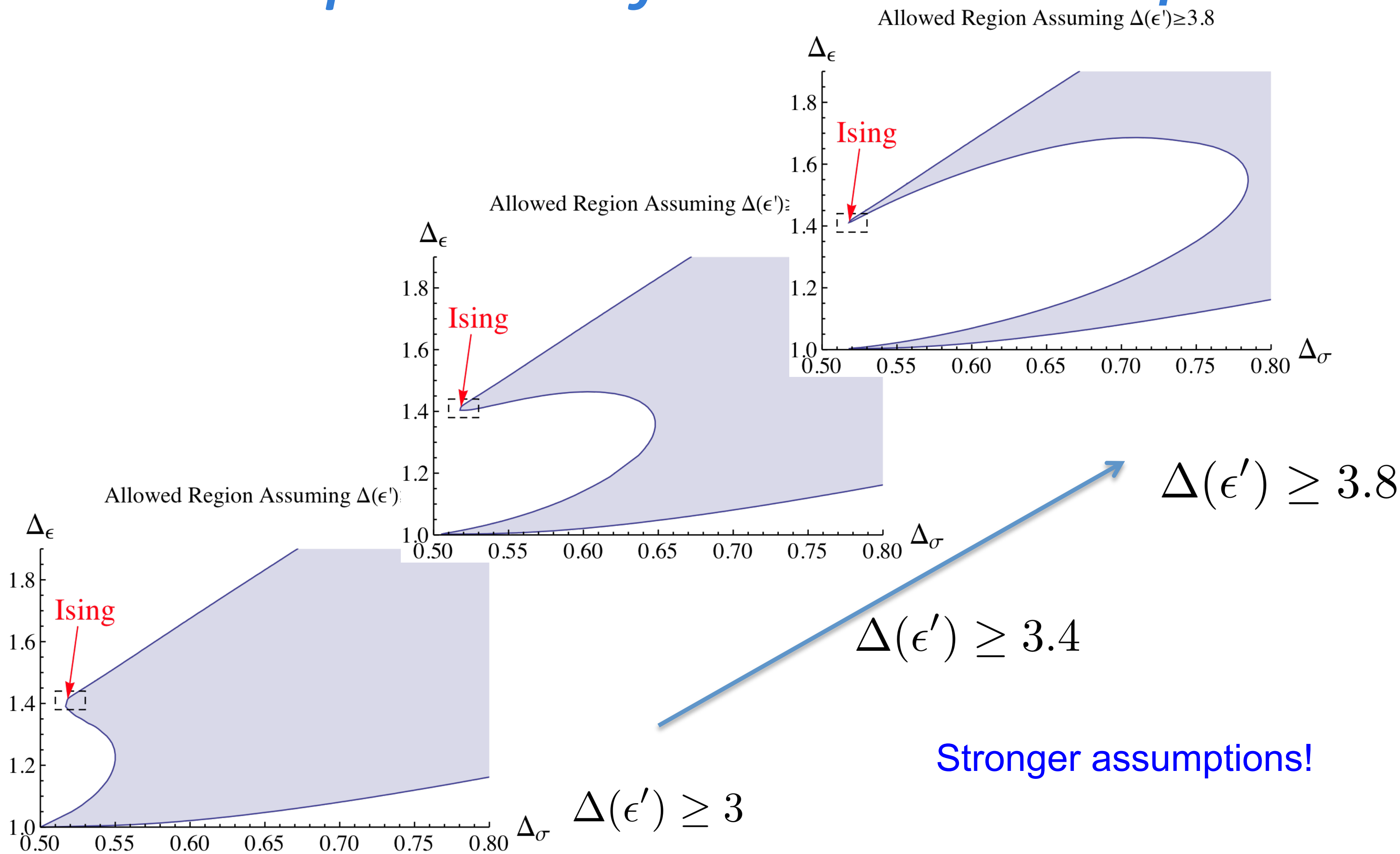
$$\mathcal{O}_i(x_1)\mathcal{O}_j(x_2) = \sum_k \frac{f_{ijk}}{|x_1-x_2|^{\Delta_i+\Delta_j-\Delta_k}}\mathcal{O}_k(0)$$

Only "tree" diagrams!
"partial waves" exp: sum over conformal blocks



CFT Bootstrap: OPE & factorization completely fixed the theory

*Inequalities from Bootstrap**



•"Solving the 3D Ising Model with the Conformal Bootstrap" (El-Showk, Paulos, Poland, Rychkov, Simmons-Duffin and Vichi) arXiv:1203.6064v1v [hep-th] (2012)

Narrative

I. First attempt

-- *Lattice Radial Quantization: 3D Ising*

R.C.B., G.T. Fleming and H. Neuberger, Phys. Lett. B 721 (2013)

II. What worked and what failed.

III. Finite Elements Methods (FEM) to the rescue ?

IV. Future hopes and dreams

Contents lists available at SciVerse ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Lattice radial quantization: 3D Ising

R.C. Brower[a,*], G.T. Fleming[b], H. Neuberger[c,1]

[a] Department of Physics, Boston University, Boston, MA 02215, USA
[b] Department of Physics, Yale University, New Haven, CT 06520, USA
[c] Department of Physics and Astronomy, Rutgers University, Piscataway, NJ 08855, USA

ABSTRACT

Lattice radial quantization is introduced as a nonperturbative method intended to numerically solve Euclidean conformal field theories that can be realized as fixed points of known Lagrangians. As an example, we employ a lattice shaped as a cylinder with a 2D Icosahedral cross-section to discretize dilatations in the 3D Ising model. Using the integer spacing of the anomalous dimensions of the first two descendants ($l = 1, 2$), we obtain an estimate for $\eta = 0.034(10)$. We also observed small deviations from integer spacing for the 3rd descendant, which suggests that a further improvement of our radial lattice action will be required to guarantee conformal symmetry at the Wilson–Fisher fixed point in the continuum limit.

Lesson #1:
Radial Quantization is not necessarily Conformal

- Consider large N 2d O(N) sigma model:

  (what happens to conformal anomaly?)

$$\mathbb{R}^2 \to \mathbb{R} \times \mathbb{S}^1$$

-- Quantize on R^2: Get Lorentz invariant theory

-- Quantize on RxS1: Get Dilation invariant

But NOT both!

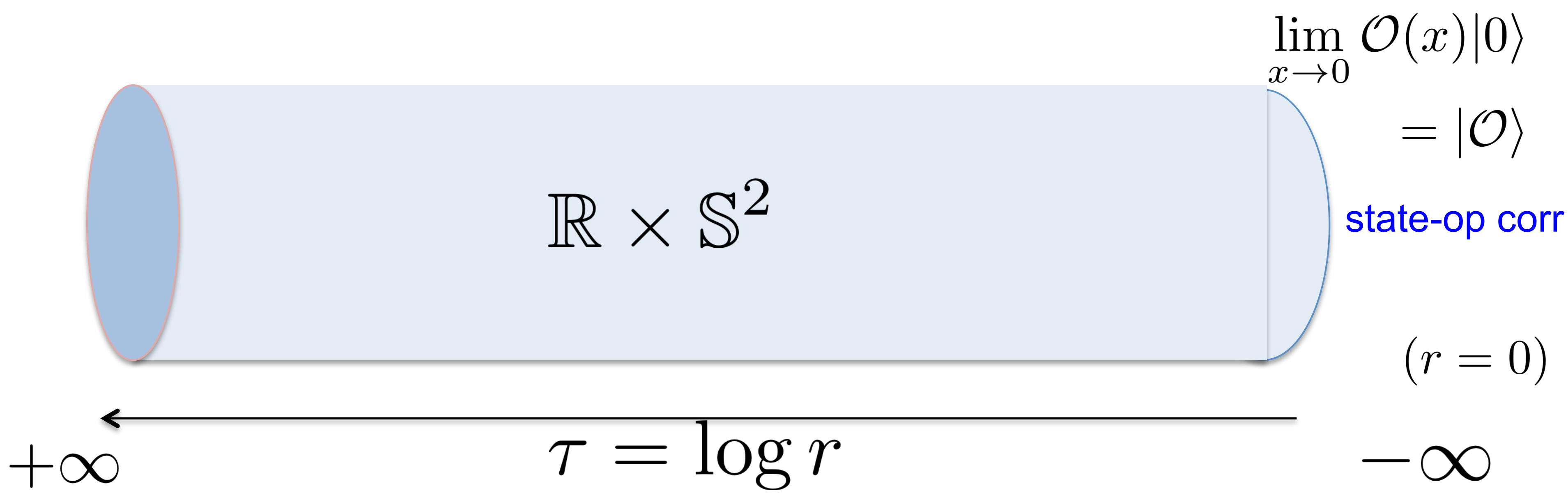
What happens to Radial Quantization? Try it?

$$\langle \Phi^{*\,i}_{\,l'}(\tau)\Phi^j_l(\tau')\rangle = \delta^{ij}\delta_{ll'}\frac{e^{-\sqrt{l^2+\mu^2}|\tau-\tau'|}}{2\sqrt{l^2+\mu^2}} \ .$$

LESSON: Descendants don't have integer-spaced descendants. Consequently, we cannot construct translation generators satisfying the correct commutation relations with dilatations in the sector generated by the action of $\vec{\Phi}$ on the vacuum. The deviation of the dilatation spectrum from equal spacing is small if $l \gg \mu$. Because inversion has also been preserved in the quantization, if translations could be realized, special conformal transformations would come in automatically and the full conformal group would be realized. Because of rotation invariance, only one linear combination of translations needs to be considered in detail.

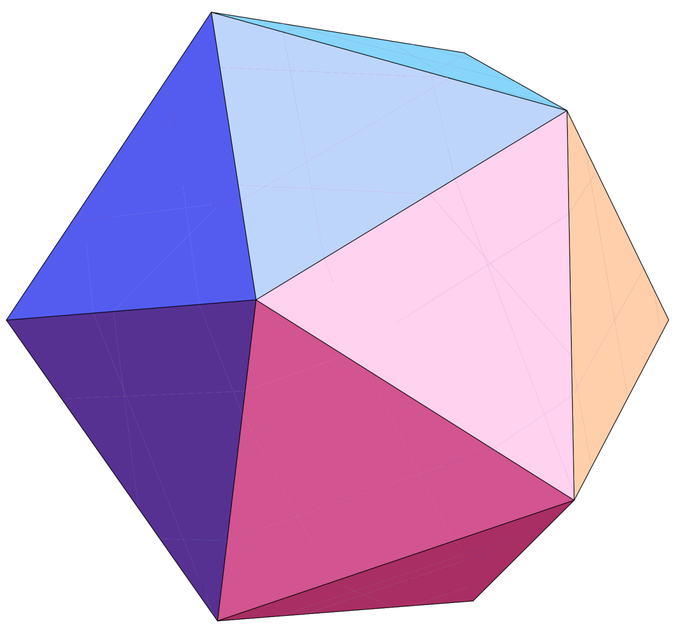
Lesson #2:
3-d Ising at Wilson-Fisher FP

$$Z_{Ising} = \sum_{\sigma(x,t)=\pm 1} e^{\beta \sum_{t,\langle x,y\rangle} \sigma(t,x)\sigma(t,y) + \beta \sum_{t,x} \sigma(t+1,x)\sigma(t,x)}$$

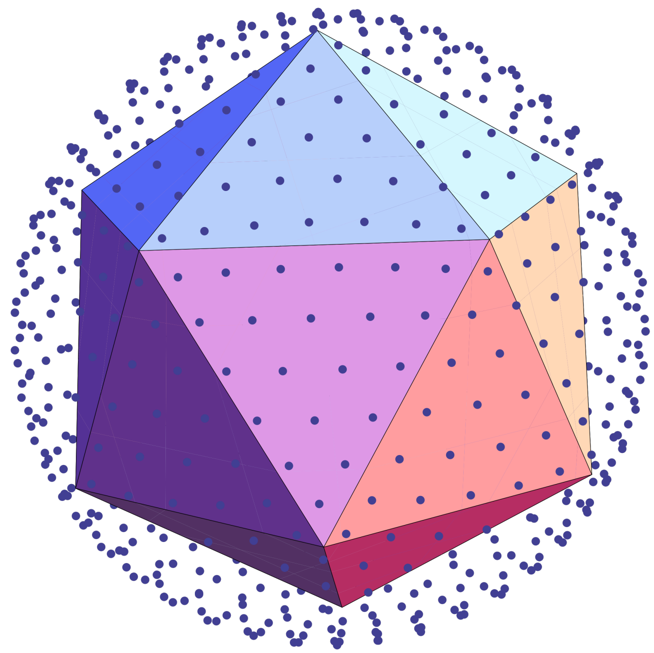


Order s Refined Triangulated Icosahedron

$s = 1$

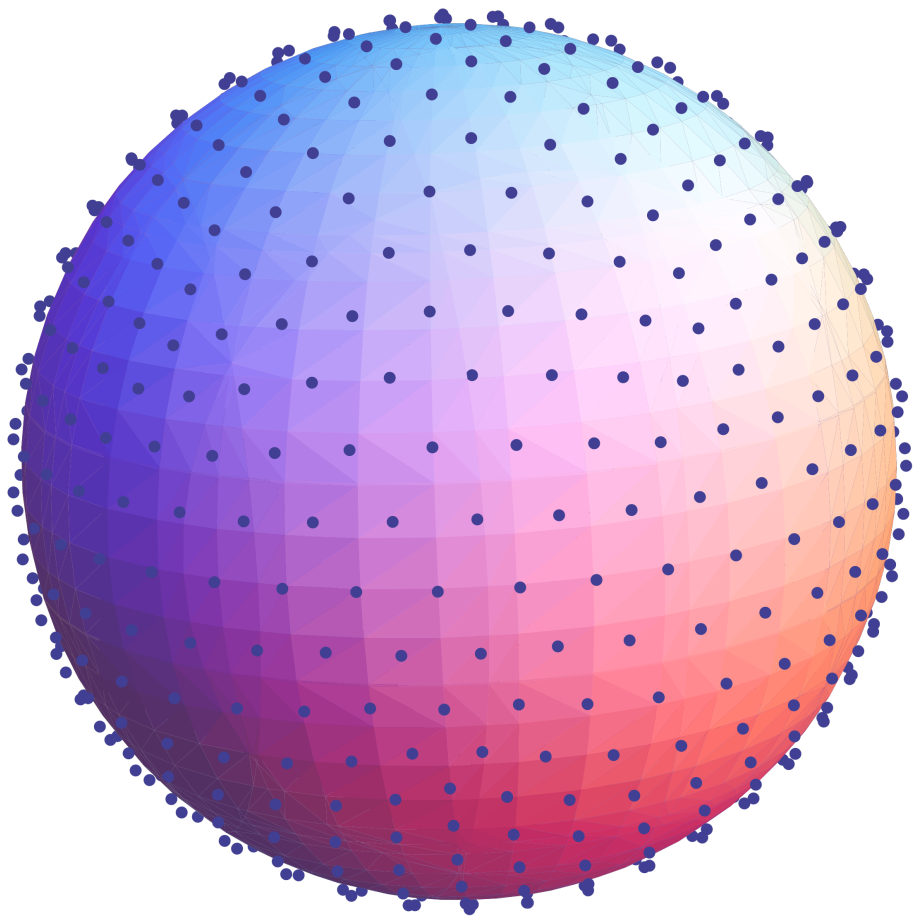


$s = 8$



l = 0 (A),1 (T1) , 2 (H) are irreducible 120 Iscosahedral subgroup of O(3)

Fixed t lattice are s refined Icosahedrons



s = 8

vertices:
N = 2+ 10*s*s = 138

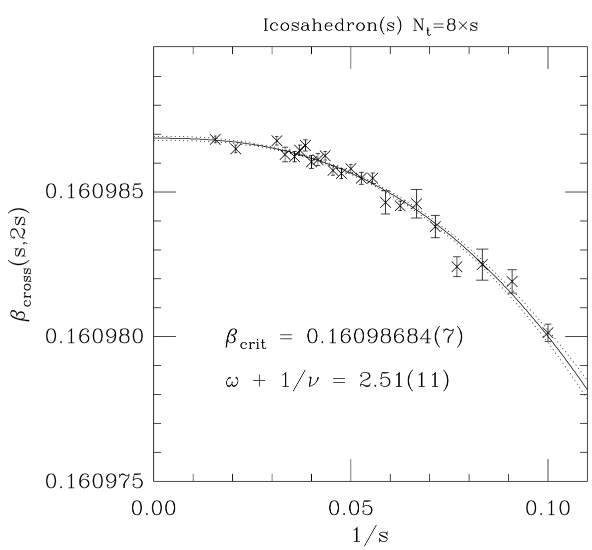
edges:
E = 3*N – 6

faces:
F = E – N + 2 = 2*N - 4

Continuum limit is $s \to \infty$ at $\beta = \beta_{critical}$

Fitting to Finite scaling

$$U[(\beta-\beta_{cr})L^{1/\nu},(\lambda-\lambda_{cr})L^{-\omega},...]\simeq$$
$$U^*(x)+O(L^{-\omega})\simeq U^*(0)+a_1(\beta-\beta_{cr})L^{1/\nu}+c(\lambda)L^{-\omega}+\cdots$$



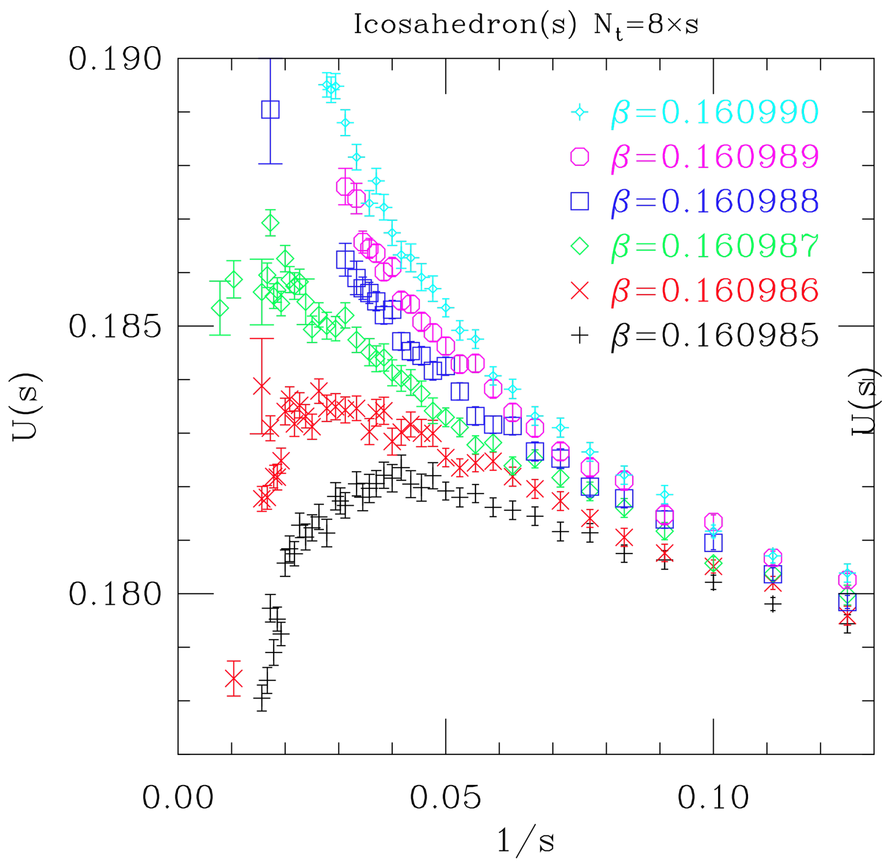
$$U(\beta,s=L)=1-\frac{\langle M^4\rangle}{3\langle M^2\rangle^2}$$

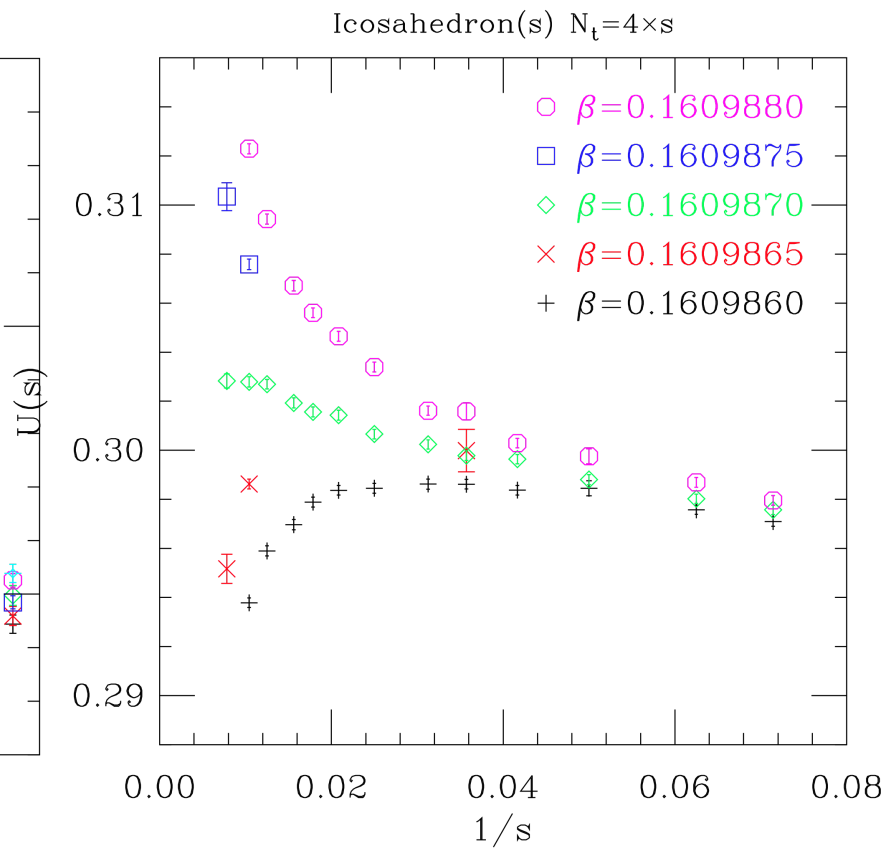
$$\beta_{cross}\simeq\beta_{cr}+c_1L^{-1/\nu-\omega}$$

$$\beta_{crit}=0.16098703(3)$$

Double Scaling: $x=(\beta-\beta_{cr})L^{1/\nu}$

Determining beta_critical





$$U(\beta, s = L) = 1 - \frac{\langle M^4 \rangle}{3\langle M^2 \rangle^2}$$

$$\beta_{crit} = 0.16098703(3)$$

Improved Operators

$$\widetilde{\sigma}_{lm}(t) = \sum_x \sqrt{\omega_x}\, Y_{lm}(\theta_x, \phi_x)\sigma(t,x)$$

Area of triangle projected on the units sphere

$$\lim_{s\to\infty} \sum_{x=1}^{2+10s^2} \omega_x\, Y^*_{l'm'}(\theta_x,\phi_x)Y_{lm}(\theta_x,\phi_x) = 4\pi\delta_{l'l}\delta_{m'm}$$

Orthonormality
to 10^{-3}, $4 * 10^{-5}$, $5 * 10^{-6}$
at $s = 16, 64, 256$ respectively

Observables

$$C_{lm}(t) = \sum_{t_0,x,y} \sqrt{\omega_x \omega_y}\, Y^*(\hat{x})\, \langle \sigma(t+t_0,x)\sigma(t_0,y)\rangle\, Y_{lm}(\hat{y})$$

cosh fit:

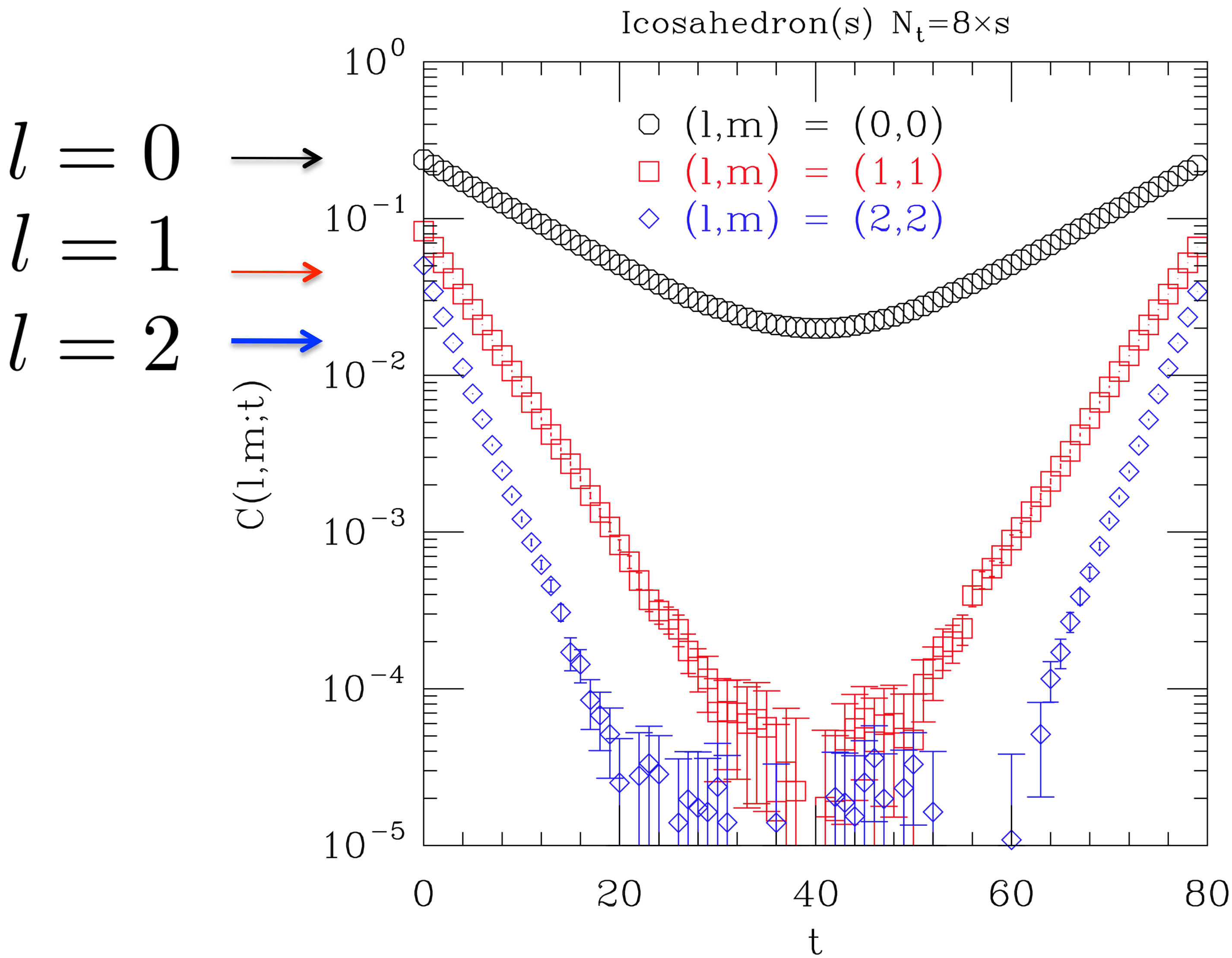
$$C_{lm}(t) = C[e^{-m_l t} + e^{-m_l(N_t-1-t)}]$$

for $t = 0, \cdots, N_t - 1\ ,\ l = 0, 1, 2\ ,\ m = -l, \cdots, l$

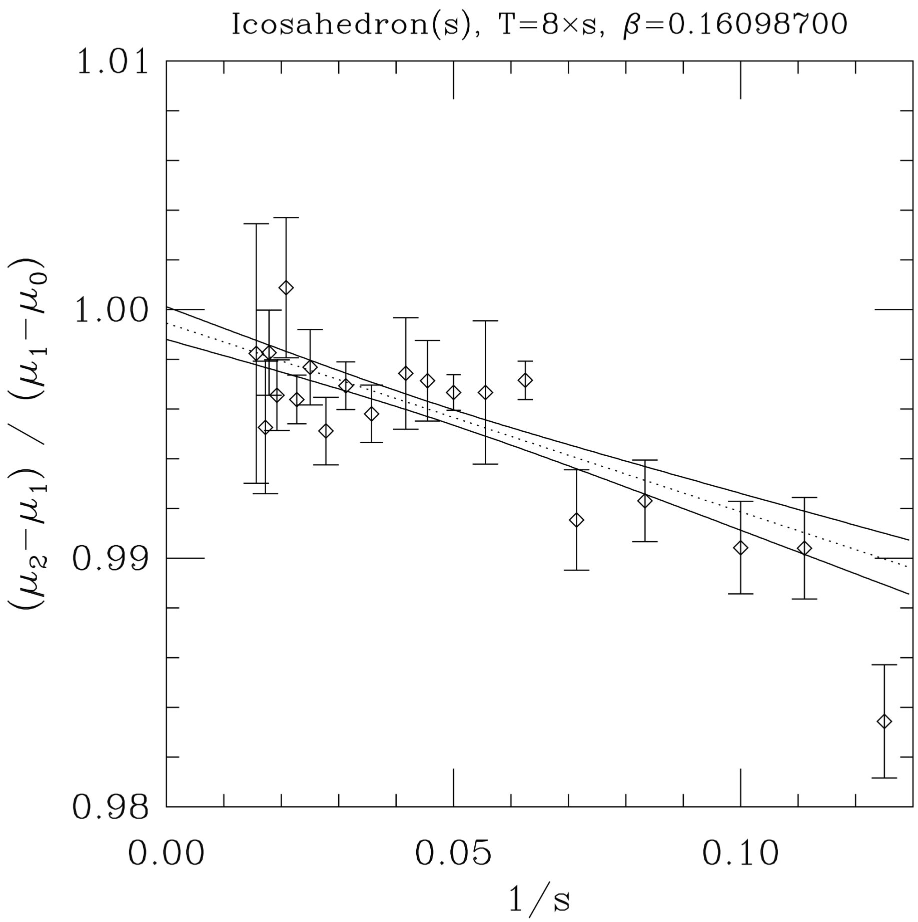
$$m_l = \frac{c}{s}\Delta_l \qquad \Delta_l = \frac{1}{2} + \frac{\eta}{2} + l$$

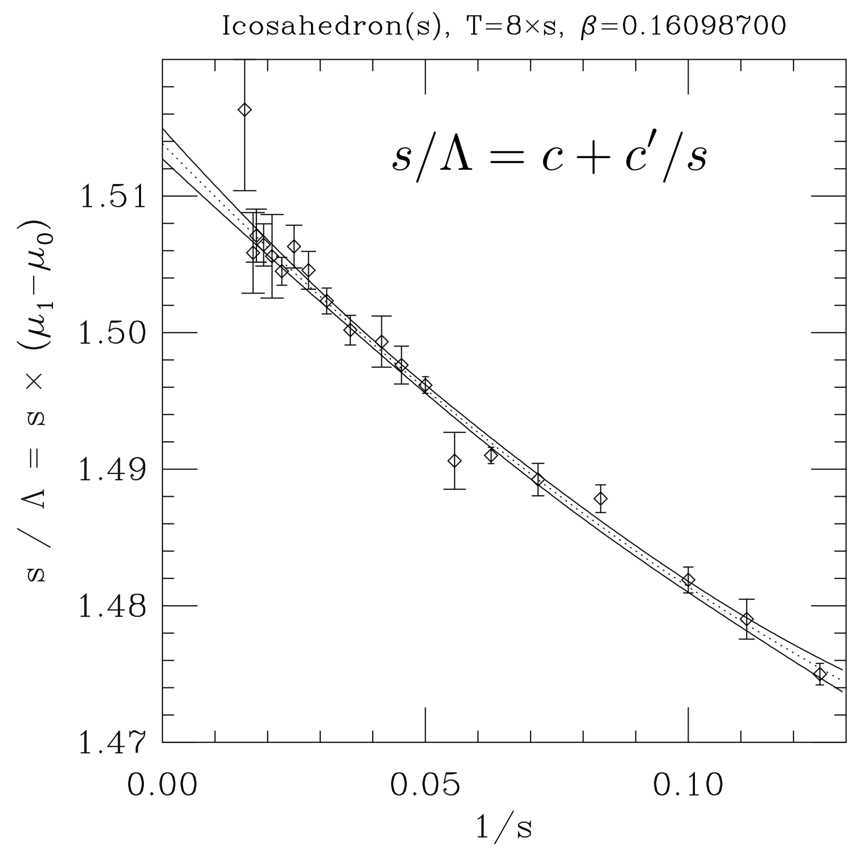
After you adjust c = `speed of light` so $\Delta_{l+1} - \Delta_l = 1$

Early result for C(t)



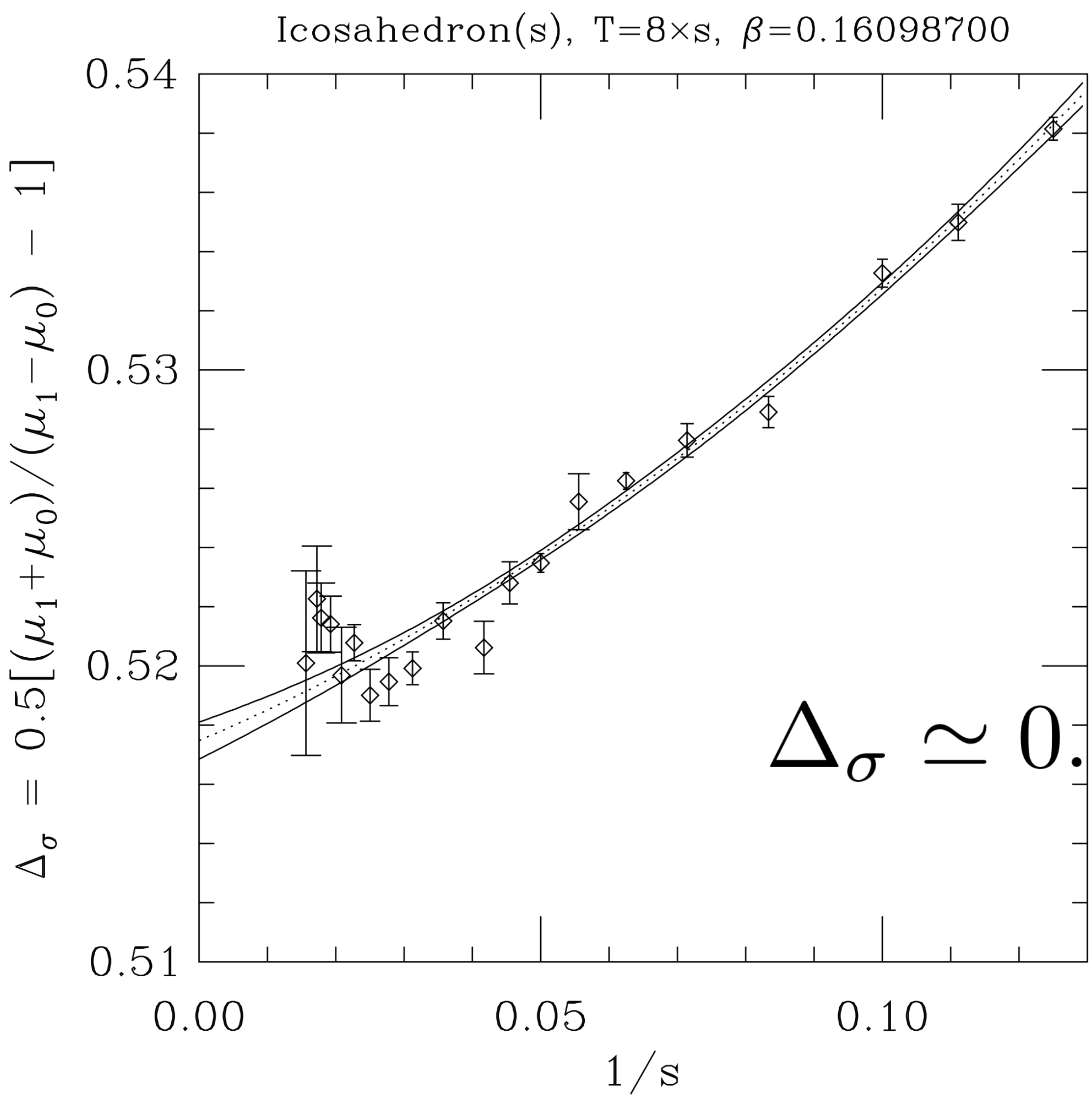
Check Descendant Relation & rescale "log(r)"





$$c = 1.5105(7)$$

Current Fit:



$$\Delta_\sigma \simeq 0.5175$$

Improved cluster Estimator

Swendsen-Wang: Real space

$$g(x-y) = \langle s_x s_y \rangle \simeq \frac{1}{N_{config}} \sum_{i=1}^{N_{config}} \sum_{C_i} \Delta_{C_i}(x) \Delta_{C_i}(y)$$

$$\Delta_C(x) = \quad 1 \text{ if } x \in C \text{ else } 0$$

Wolff single cluster

$$\widetilde{g}_{lm}(k) \simeq \frac{1}{N_{config}} \sum_{i=1}^{N_{config}} \frac{1}{|C|} \Big| \sum_{t,x \in C} e^{i2\pi kt/L_t} Y_{lm}(\Omega_x) \Big|^2$$

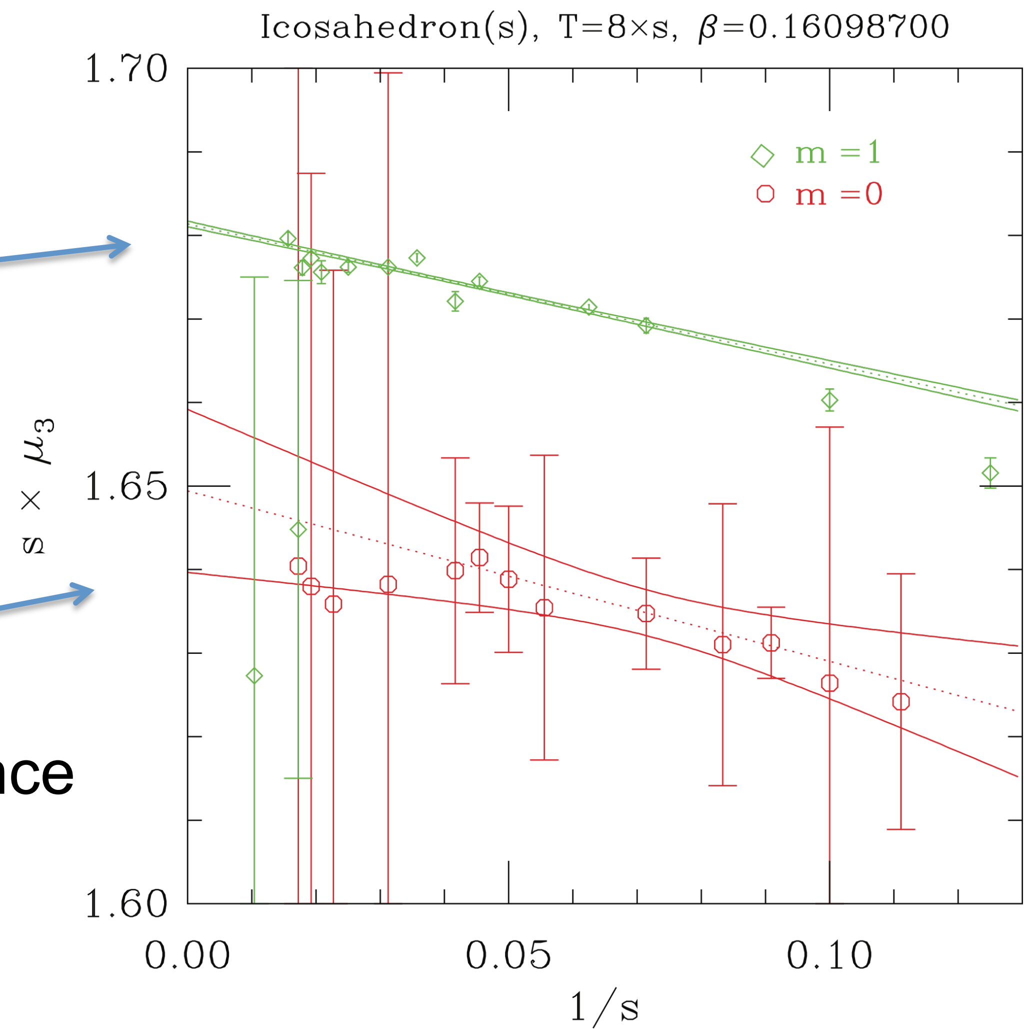
Note: All to All O(V) improved estimator in Momentum space *

*C. Ruge, P. Zhu and F. Wagner Physica A (1994) 431:

Wrong Theory?
Failure to recover O(4,1) of l = 3?

G rep

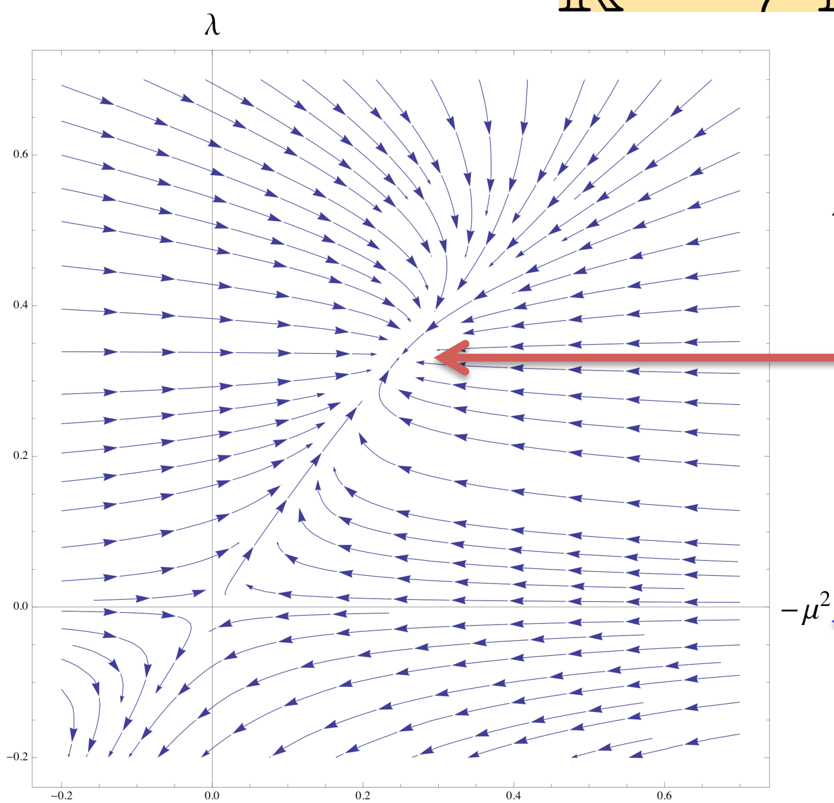
T2 rep



Apparent lack of convergence to a single O(3) irreducible representation for l = 3

Replace Ising Model by phi 4th

$$\mathbb{R}^d \to \mathbb{R} \times \mathbb{S}^{d-1}$$



$$\mathcal{L}(x) = -\frac{1}{2}(\nabla\phi)^2 + \lambda(\phi^2 - \mu/2\lambda)^2$$

Wilson-Fisher FP

Gaussian FP

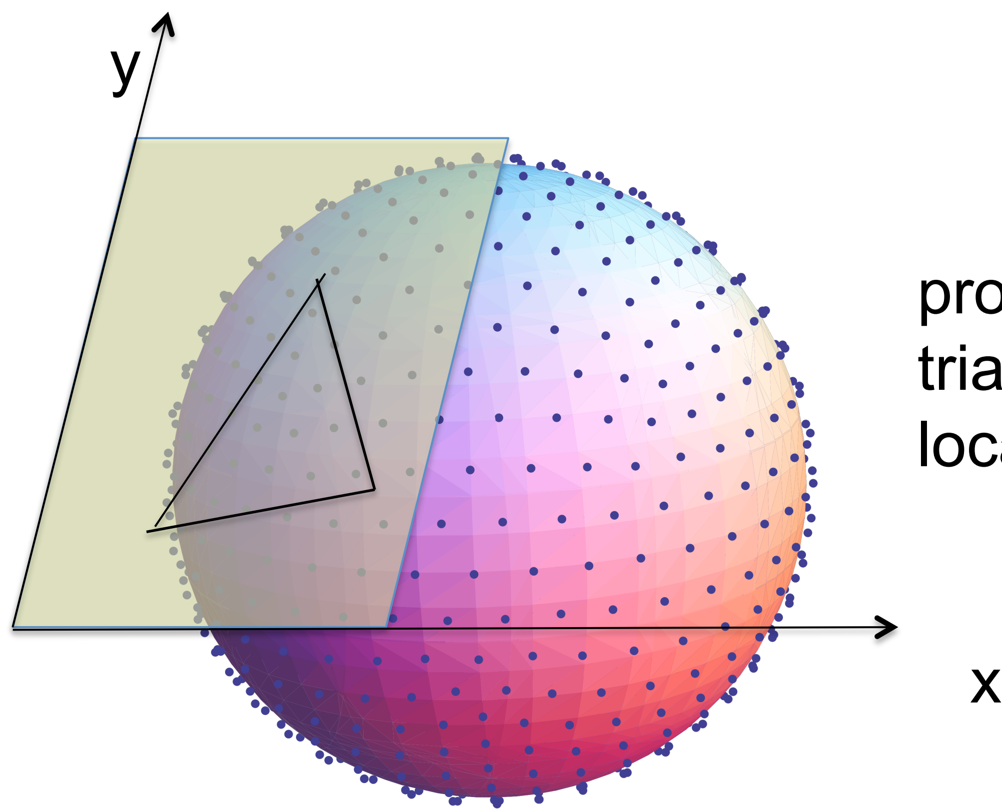
$$\beta_g = \epsilon g - \frac{3}{16\pi^2}g^2 + O(g^3, \epsilon g^2, \mu^4, \mu^2 g;)$$

$$\beta_{\mu^2} = 2\mu^2 + ag + \frac{9}{16\pi^2}g\mu^2 + O(\mu^4)$$

$$\lambda = 4g/4!$$

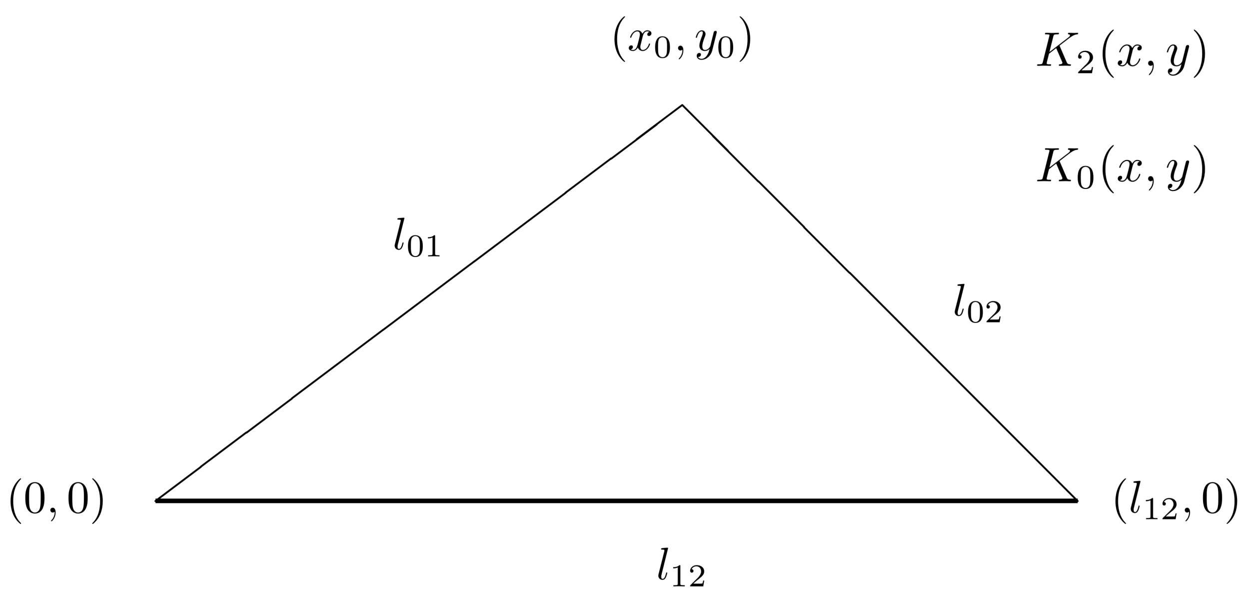
Finite element Discrete Lagrangian on triangulated sphere.

$$Z = \int \mathcal{D}\phi e^{-\frac{1}{2}K_{ij}(\phi_i - \phi_j)^2 - \lambda\omega_i(\phi^2 - \mu/2\lambda)^2}$$



project spherical
triangle onto
local tangent plane

Kinetic term for Linear Element



$$K_1(x,y) = [l_{12} - x - \frac{(l_{12} - x_0)y}{y_0}]/l_{12}$$
$$K_2(x,y) = [x - \frac{x_0 y}{y_0}]/l_{12}$$
$$K_0(x,y) = \frac{y}{y_0}$$

On each triangle expand: $\phi(x,y) = \sum_i K_i(x,y)\phi_i$ an integrate

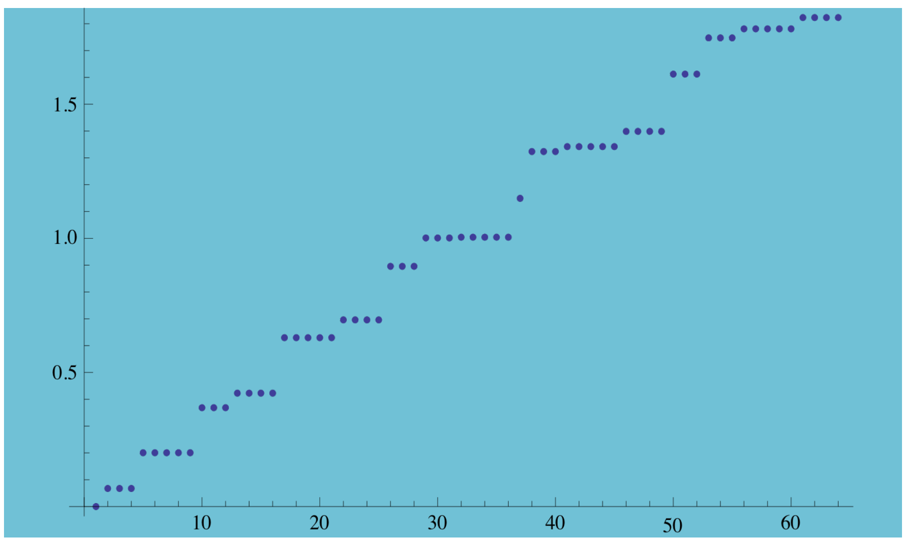
$$\int_{A_{012}} dxdy \partial_\mu \phi(x,y) \partial_\mu \phi(x,y) = \frac{1}{2A_{012}}[(l_{01}^2 + l_{20}^2 - l_{12}^2)(\phi_1 - \phi_2)^2 + \text{cyclic}]$$

See also: Christ, Friedberg, Lee ion "Random Lattice" NP (1982)

FEM fixes the huge Spectral defects

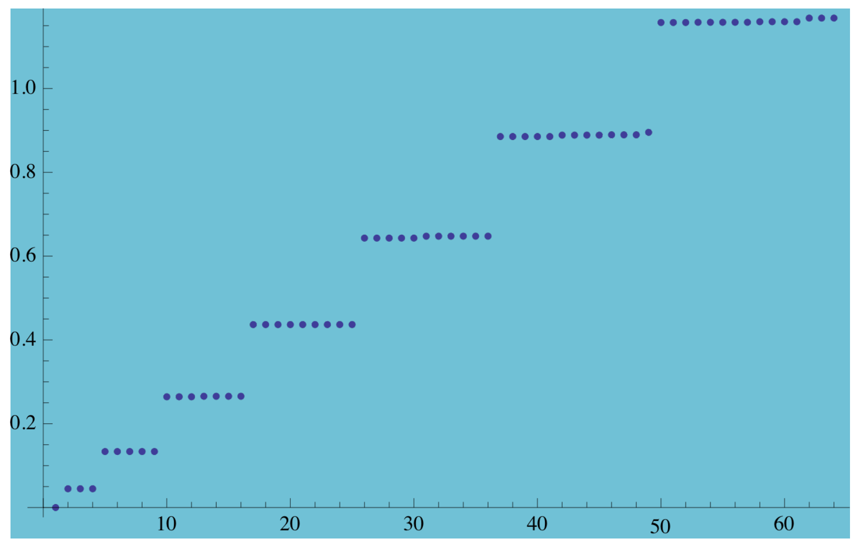
For s = 8 first (l+1)*(l +1) = 64 ev

BEFORE (K = 1)



l, m

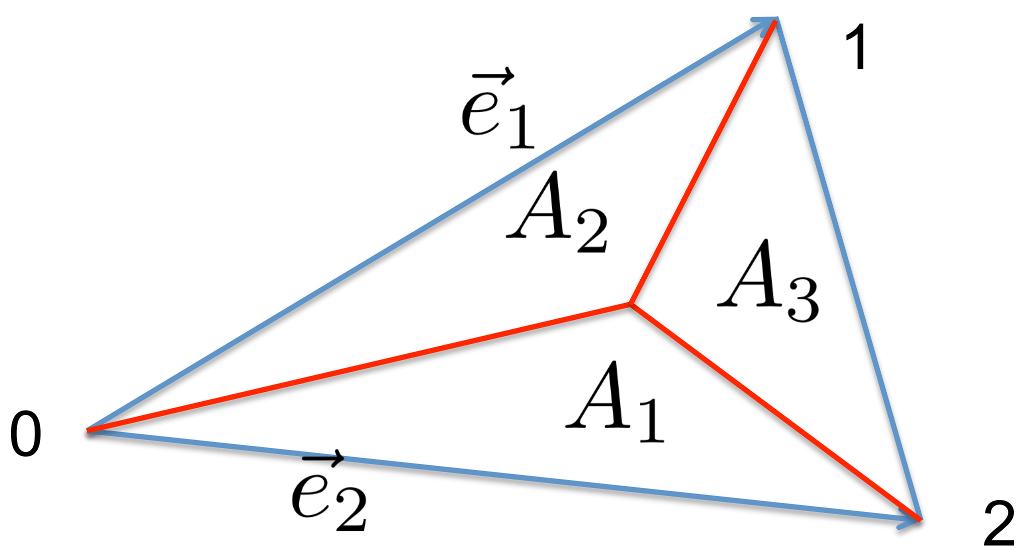
AFTER (FEM K's)



l, m

Linear Finite Element Method for triangulate Manifold

$$g_{ij}(0) = \vec{e}_i \cdot \vec{e}_j$$



$$K_i^{\triangle}(r) = \frac{A_i}{A_{123}}$$

$$W_i(r) = \sum_{\triangle} K_i^{\triangle}(r)$$

$$\phi(r) = \sum_i W_i(r)\phi_i$$

Piecewise linear subspace of Hilbert space

$$d\vec{x} = \vec{e}_i dx^i \qquad ds^2 = d\vec{x} \cdot d\vec{x} = g_{ij} dx^i dx^j$$

FEM have “spectral fidelity”

- Taylor expansion on hypercubic lattice:

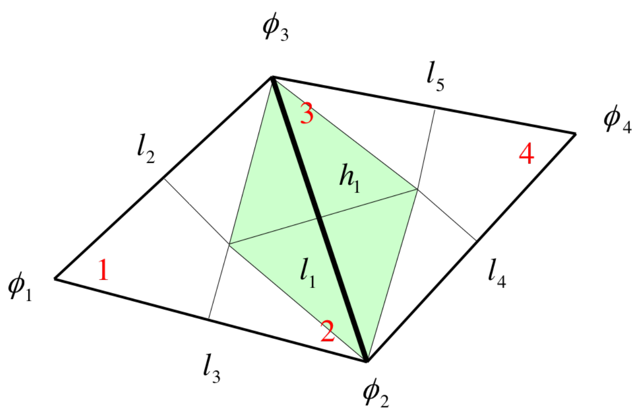
$$a^{-1}\sum_{\pm\mu}(\phi(x)-\phi(x+a\mu)^2 \simeq (\nabla\phi)^2 + O(a^2)$$

- Taylor series for FEM does not work!

$$a^2\sum_{y}K(x,y)(\phi(x)-\phi(y))^2 \simeq c_{\mu\nu}\partial_\mu\phi\partial_\nu\phi + O(a^2)$$

- FEM “meta-theorem”: spectrum < cut-off to O(a^2) if “triangles are regular enough”

Geometrical Interpretation of kinetic term:
Essentially identical to Regge Calculus on smooth manifold.



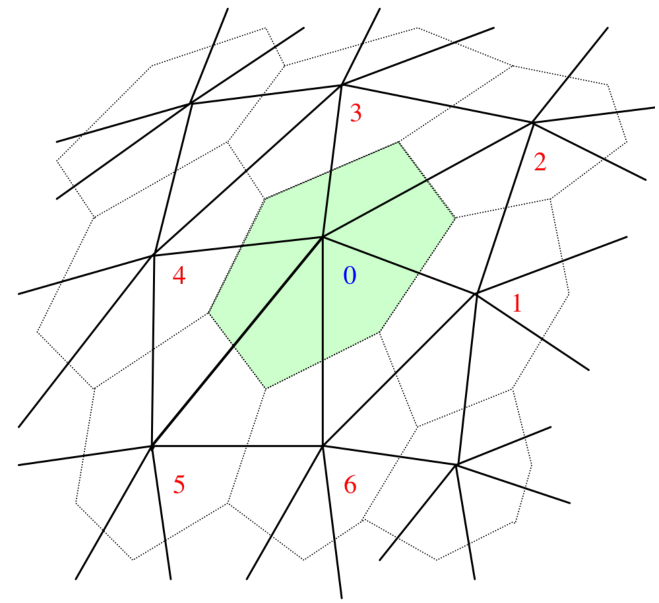
$$FEM : A_d \frac{(\phi_2 - \phi_3)^2}{l_1^2}$$

$$\text{Delaunay Link Area:} \quad A_d = h_1 l_1$$

$$\sum_{\triangle_{kij}} \sqrt{g(k)}\, g^{ij}(k) \frac{(\phi_k - \phi_i)(\phi_k - \phi_j)}{l_{ki} l_{kj}}$$

H. Hamber, S. Liu, *Feynman rules for simplicial gravity,* NP B475 (1996)

Einstein Regge Curvature



$$\delta_v = 2\pi - \sum_{i \in V} \theta_i$$

$$\sum_v \delta_v = 2\pi\chi = 2\pi(F - E + V)$$

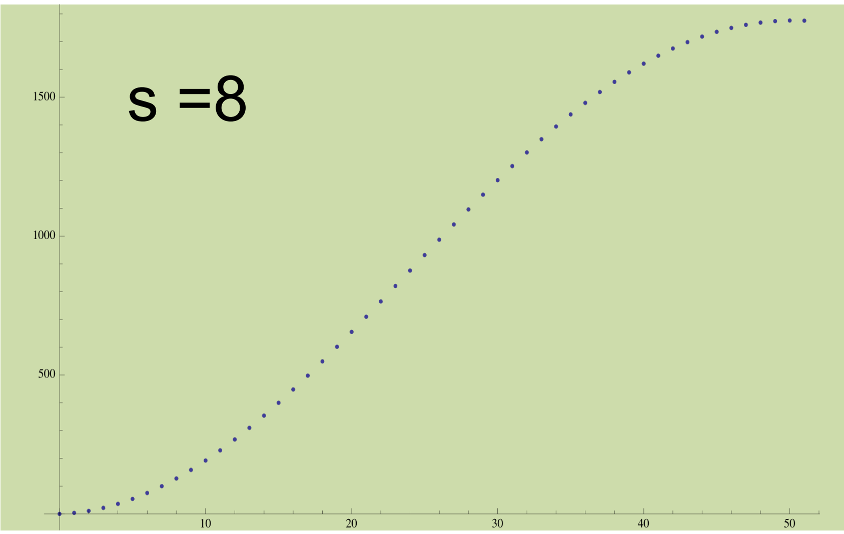
Could Optimize adaptive Delaney triangles on unit sphere

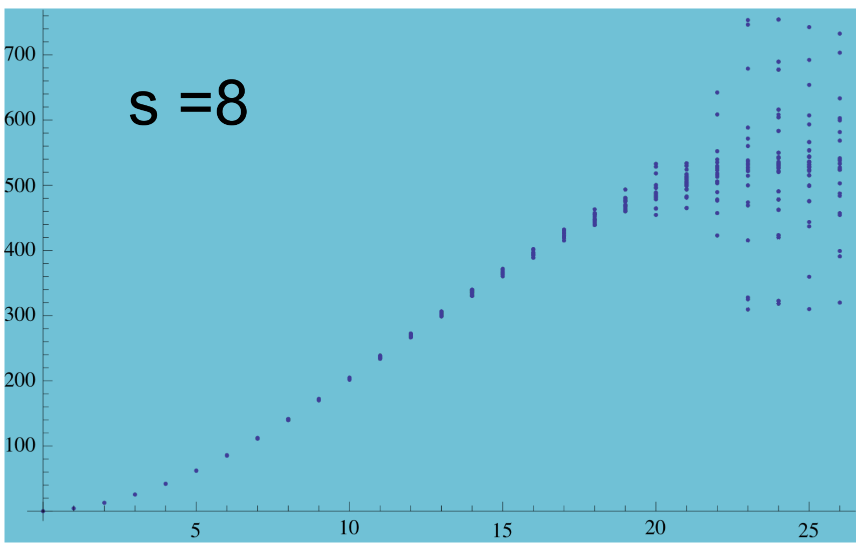
$$\int d^2x \sqrt{g}[\lambda - kR^2 + aR^2] \implies \sum_v A_v[\lambda - 2kR_v + aR_v^2]$$

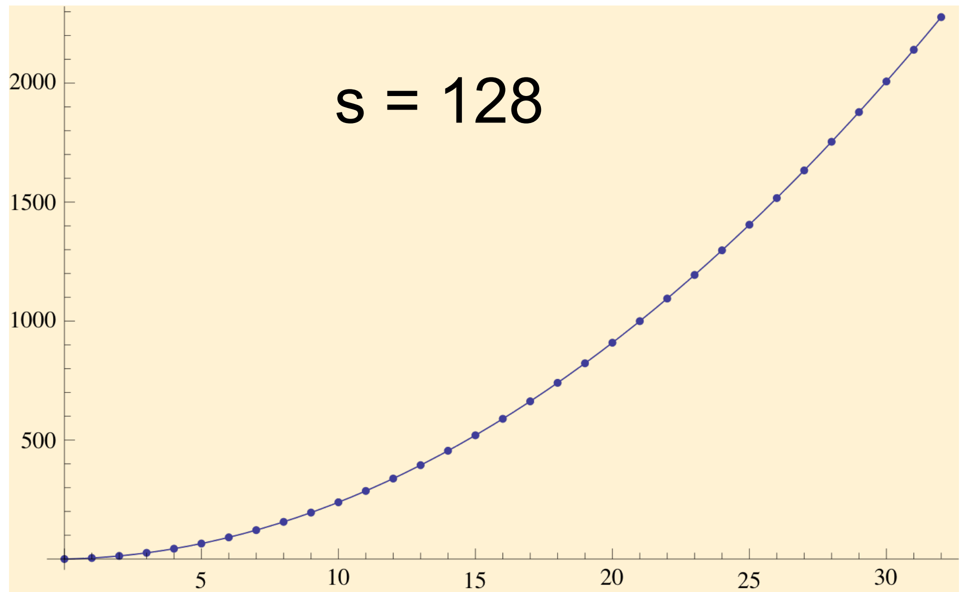
$$R_v = 2\delta_v / A_v$$

flat triangles: $\delta_v = 4\pi / A_v$

Spectrum of FE Laplacian on a sphere



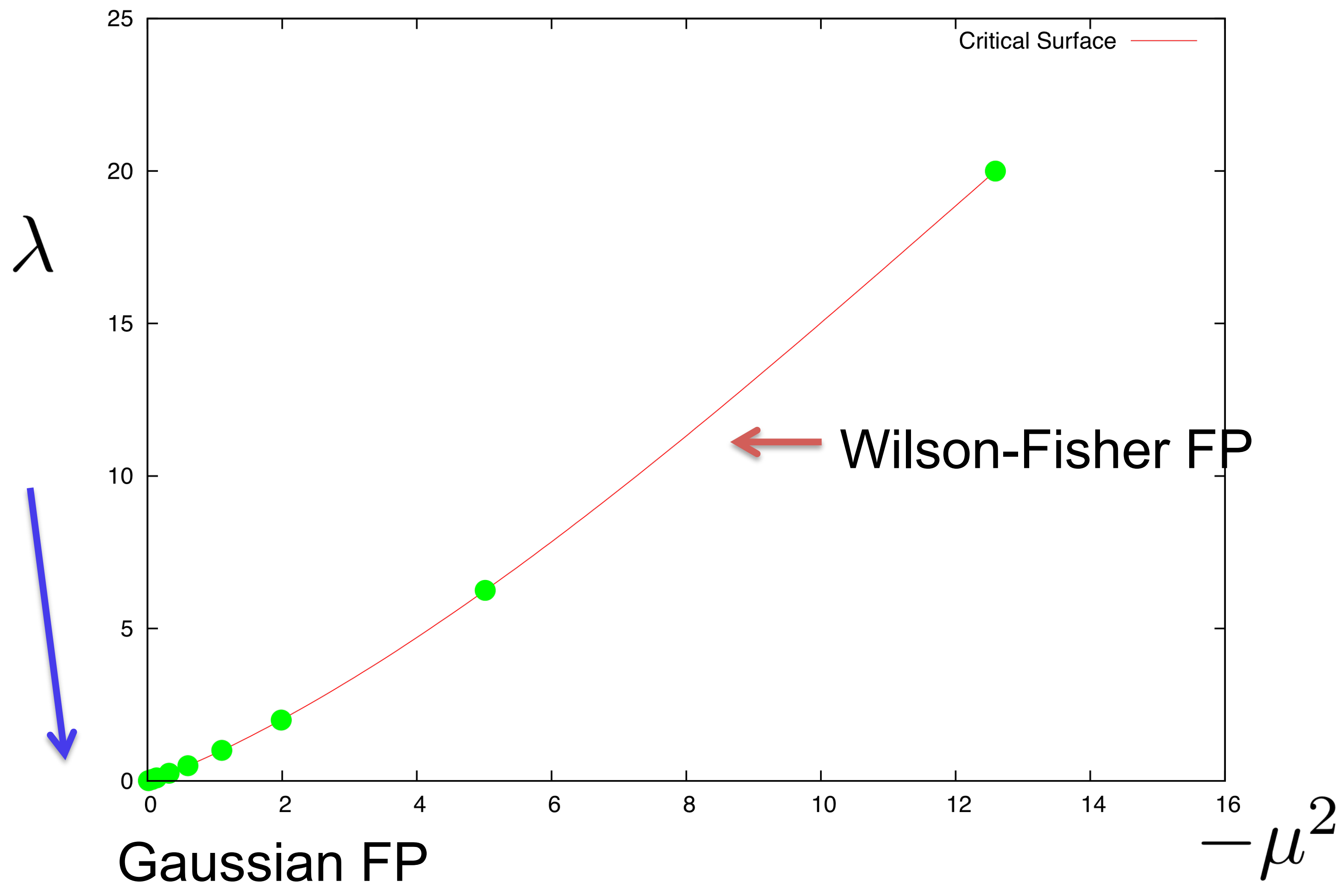




Fit

$$l + 1.00012\, l^2$$
$$- 13.428110^{-6}\, l^3 - 5.5724410^{-6}\, l^4$$

Radial Lattice Critical Surface



The simulation program is running

(1) Monte Carlo is a “standard” mixture of metropolis, over relax and Wolff methods from:

Ulli Wolff, “Collective Monte Carlo Updating for Spin Systems PRL 62: 361 (1989)

R.C.B. and P. Tamayo, “Embedded Dynamics for phi4 Theory”, PRL 62:1087(1989)

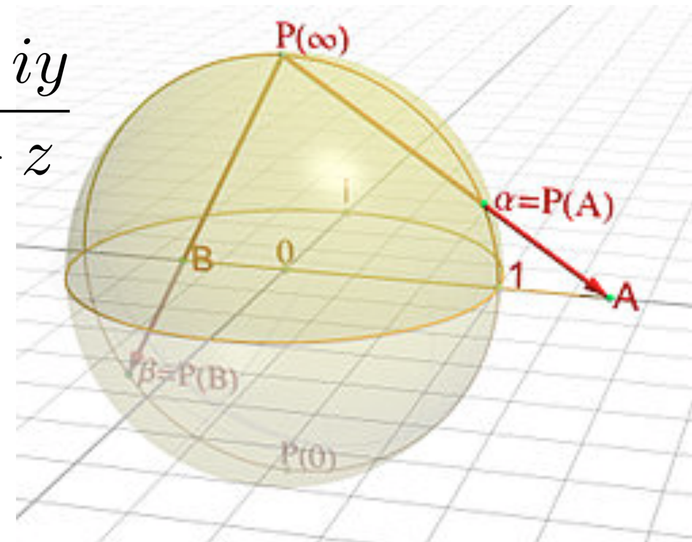
(2) Will compute higher primaries, even Z2 sector, Energy momentum tensor, Conformal Blocks partial waves

(3) The code can run any graph, so we will replace sphere by torus to reproduce phi 4^{th} numbers from Hasenbusch,..

Hasenbusch,``A Monte Carlo study of leading order scaling corrections of phi**4 theory on a three-dimensional lattice” J.Phys. A 32 (1999) 4851 *

2D test on Conformal Projection to Riemann Sphere

projection $\xi = \tan(\theta/2) e^{-i\phi} = \dfrac{x + iy}{1 + z}$



Exact Two point function

$$\langle \phi(x_1)\phi(x_2)\rangle = \frac{1}{|x_1 - x_2|^{2\Delta}} \to \frac{1}{|1 - cos\theta_{12}|^{\Delta}}$$

$$\Delta = \eta/2 = 1/8$$

$$x^2 + y^2 + z^2 = 1$$

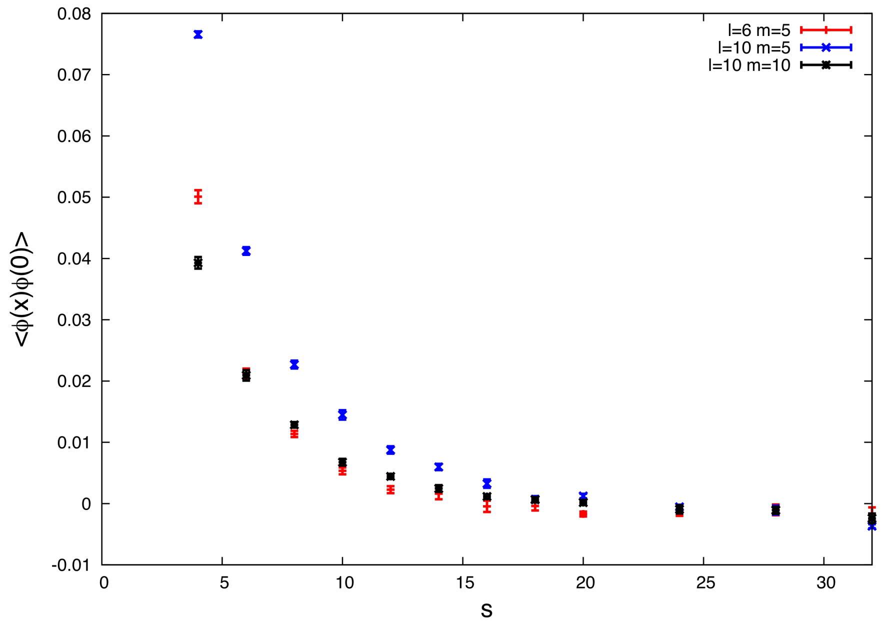
4 pt function $\quad (x_1, x_2, x_3, x_4) = (0, \xi, 1, \infty)$

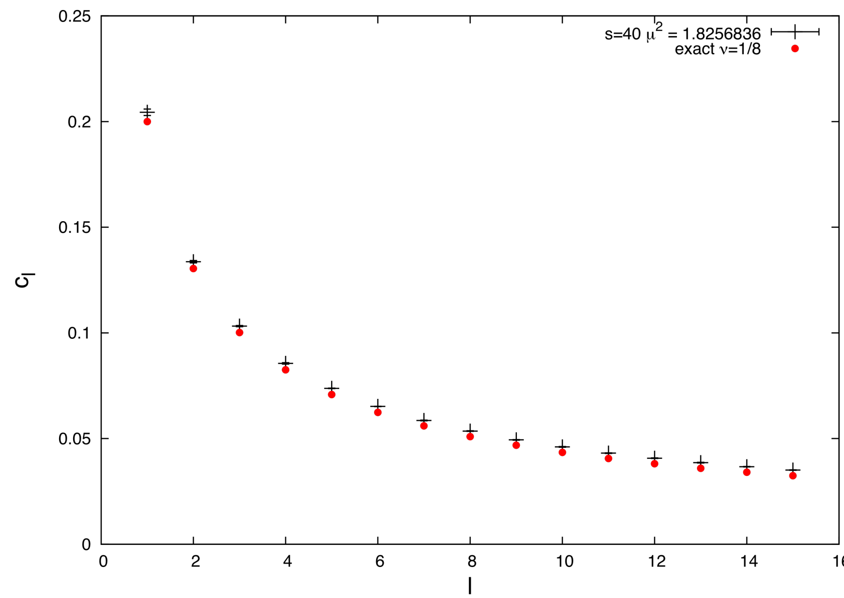
$$g(0, \xi, 1, \infty) = \frac{1}{2|\xi|^{1/4}|1 - \xi|^{1/4}}[1 + \sqrt{1 - \xi}| + |1 - \sqrt{1 - \xi}|]$$

$$U_B^* = 1 - \frac{\langle M^4 \rangle}{3\langle M^2 \rangle \langle M^2 \rangle} = 0.567336$$

Critical Binder Commulant

Test of rotational symmetry?

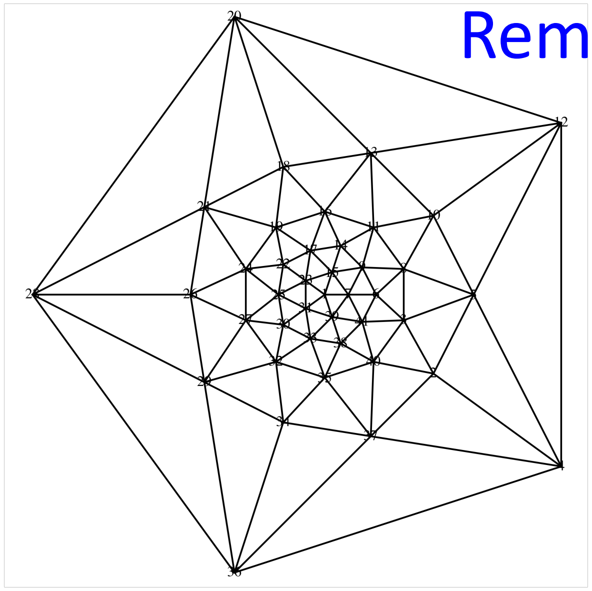




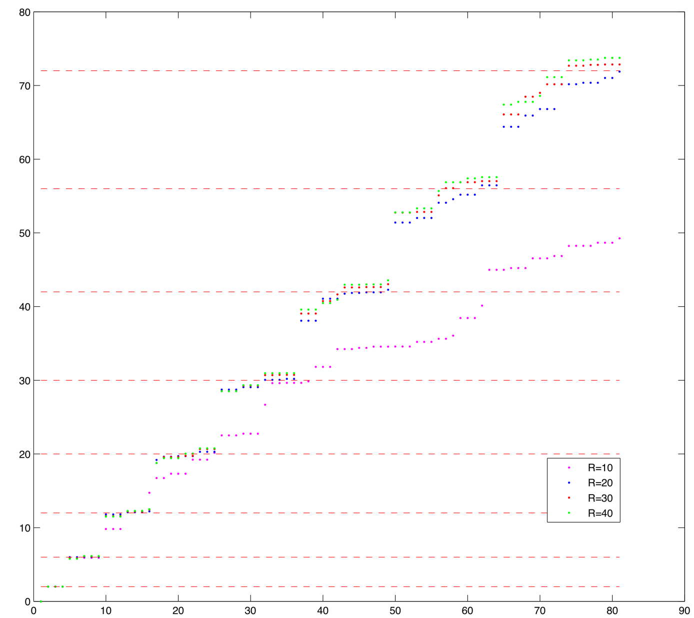
Ylm projection.

$$\Delta_\sigma = \eta/2 = 1/8 \simeq 0.128$$

Remarks on Riemann Sphere



Deluanay triangulation commutes with projection!



Spectrum on Cubic Sphere!

- Testing
  - Exact 2D Ising correlators, exponents, central charge, etc.
  - Improved spherical and/, 2nd order elements, etc
  - Dynamical R^2 Curvature Constrained Triangulation.

EXTRA SLIDES

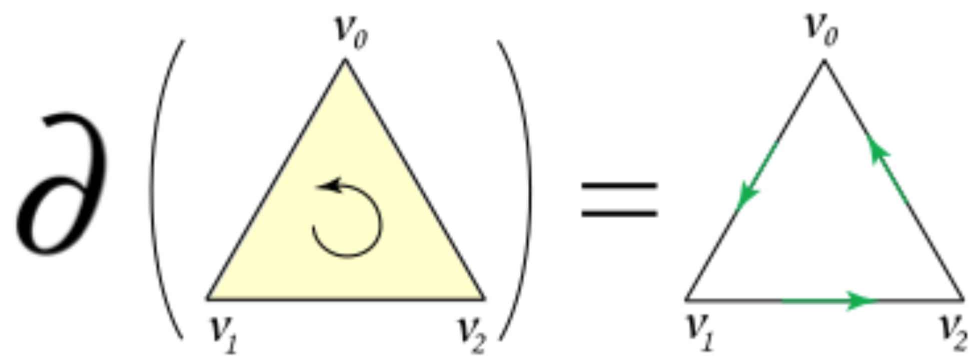
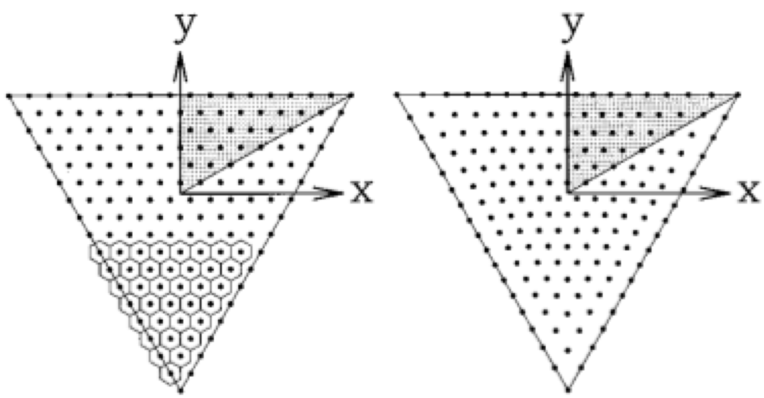
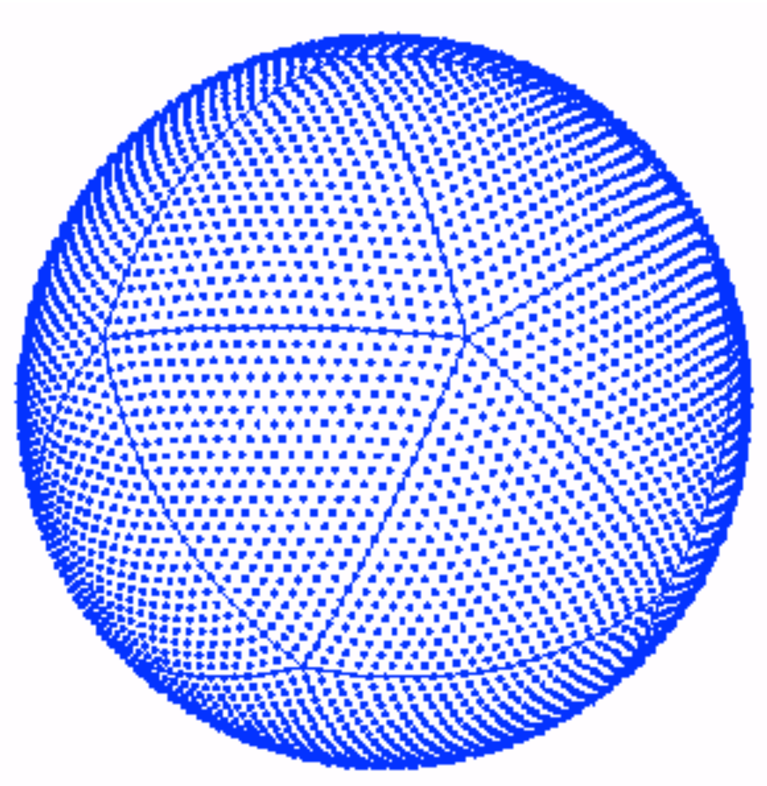


Figure 3: *The boundary operator ∂ applied to a triangle (a 2-simplex) is equal to the signed sum of the edges (i.e., the faces of the 2-simplex).*

Remarks on Correcting UV breaking?

Blackboard Discussion on UV corrections!



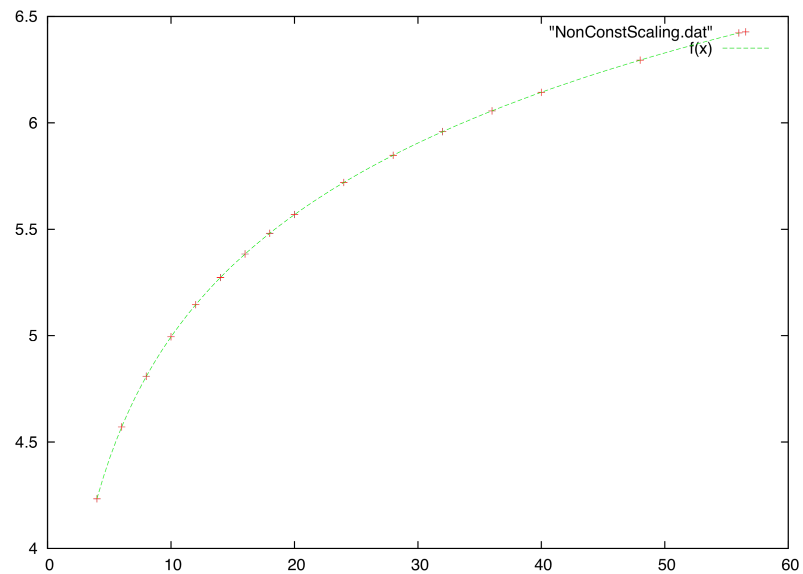


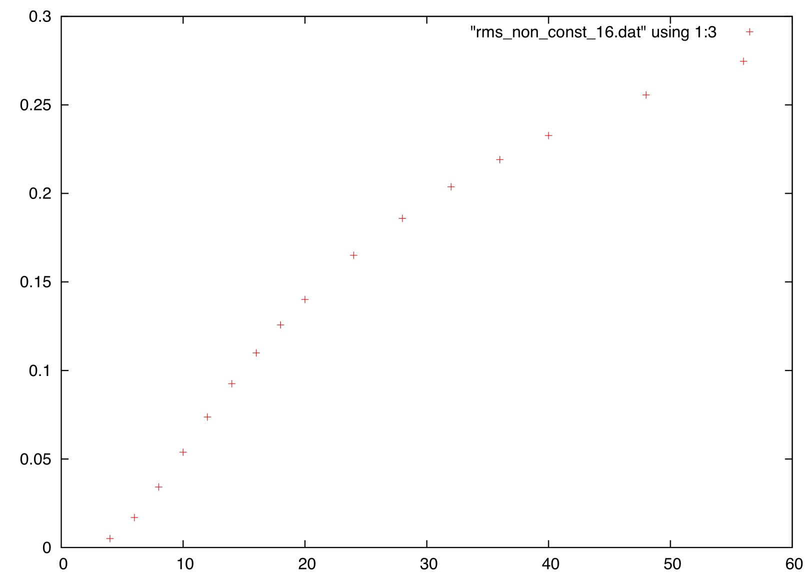
An icosahedron-based method for pixelizing the celestial sphere

MAX TEGMARK THE ASTROPHYSICAL JOURNAL, 470:L81–L84, 1996 October 20

Logarithmic Divergent One Loop

$$c(x) log(s/\mu_0)$$





$$\mu_0^2 = 1/4$$

$$\mu_0^2 = L^2 = 16^2$$

$$M(l) = l(l+1) + \mu_0^2 + ...$$

Future Challenges & Directions

- Many extensions are interesting
  - Easier problems:
    - Prove radial quantization for O(N) at large N
    - Strengthen bootstrap inequalities for spin systems?
  - Harder Problems:
    - Gauge fields (with discrete Chirstoffel connection)?
    - Fermions (with discrete spin connection) ?
    - Flow from UV to conformal IR fixed points for BSM? (Cross over from UV, to Hamiltonian to Dilation spectrum)

Fancy stuff: ''FEM'' or discrete Exterior Calculus

Need a local reference tangent plane $\xi^a(x)$at x.
Introduce an ortho normal basis in the tangent space: $\vec{e}_a(x)$

$$g_{\mu\nu}(x) = e^a_\mu(x) e^a_\nu(x) \qquad \bar{\psi} e^\mu_a \gamma^a D_\mu \psi$$

Lattice Fermions are on simplicial complex lattice manifolds with great care! Spin connection has be done carefully.

Compact gauge links can be represented also
a la Christ, Friedberg, Lee! In weak field limit maybe equivalent to using Nedelic/Whitney "edge" elements etc.

**Simplicial differential form, deRham complex a la Regge Calculus!*

Fitting correlators

- Discrete states have exact cosh correlators

$$C_l(t) = A_l \cosh(-\mu_l(t - T/2))$$

- Transform to k–space

Disconnected piece

$$\begin{aligned}\widetilde{C}_l(k) &= \frac{1}{T}\sum_{t=0}^{T-1} e^{itk}\, C_l(t) \\ &= c_0\delta_{l,0}\delta_{k,0} + a_l \frac{(1-e^{-\mu_l T})\sinh(\mu_l)}{\sinh^2(\mu_l/2)+\sin^2(k/2)}.\end{aligned}$$

Good fits required 3 mass

Primary operators 3-d Ising Model

Operator	Spin l	$\mathbb{Z}$	Δ	Exponent
s	0	$-$	0.5182(3)	$\Delta = 1/2 + \eta/2$
s'	0	$-$	$\gtrsim 4.5$	$\Delta = 3 + \omega_A$
ε	0	$+$	1.413(1)	$\Delta = 3 - 1/\nu$
ε'	0	$+$	3.84(4)	$\Delta = 3 + \omega$
ε''	0	$+$	4.67(11)	$\Delta = 3 + \omega_2$
$T_{\mu\nu}$	2	$+$	3	$\Delta = 3$
$C_{\mu\nu\kappa\lambda}$	4	$+$	5.0208(12)	$\Delta = 3 + \omega_{\text{NR}}$

Low-lying primary operators of the 3D Ising model at criticality.

Primary l = 0

$$[K_\mu, \mathcal{O}(0)] = 0$$

Descendants l > 0

$$\mathcal{O}_{l+1}(x) = [P_\mu, \mathcal{O}(x)] = i\partial_\mu \mathcal{O}_l(x)$$

Numerical Test (so far)

- Equal spacing test of descendants:

$$\frac{\mu_2 - \mu_1}{\mu_1 - \mu_0} = 0.999(1)$$

- "Speed of light" $c = 1.5105(7)$
- But critical point $\beta_{crit} = 0.16098703(3)$
- Current anomalous dimensions (more soon)
  - from Binder: $\omega + 1/\nu = 2.51(11)$
  - from corr: $\Delta_\sigma = 1/2 + \eta/2 = 0.5175(6)$
  - Simulation are on going to reduce errors