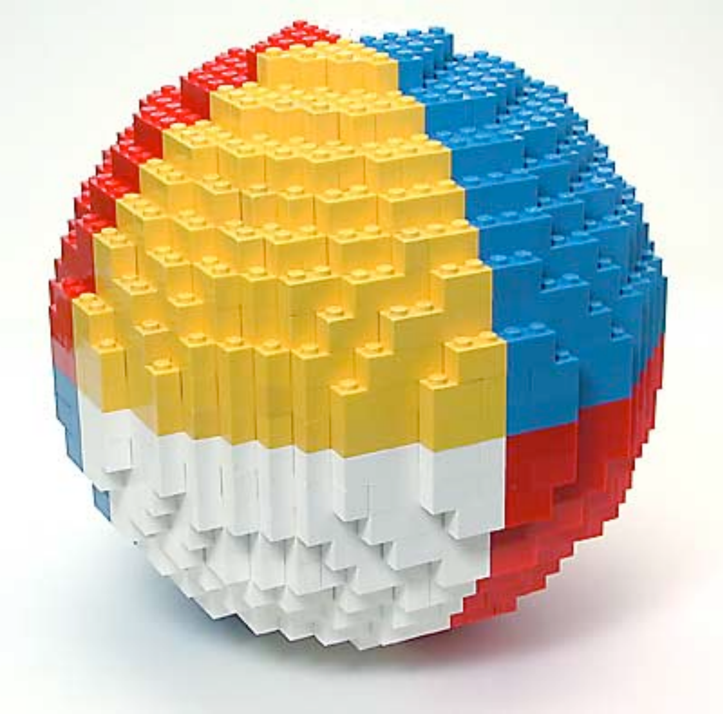
Free Scalar Fields on Spherical Lattices



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outline

- Motivation/Application
  - CFTs and Radial Lattice Quantization
- Brief Background
  - Naïve discretization; Regge calculus
- Sphere-like Lattices
  - Refined cube
  - Refined icosahedron
  - Graph construction; building the lattice action
- Convergence of the Free Scalar Spectrum
  - On the refined cube
  - On the refined icosahedron
- Future Work

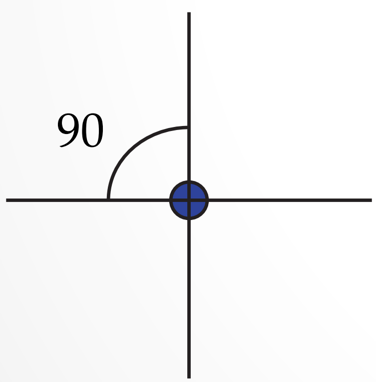
motivation

- Want to study conformal QFTs on the lattice
- Can map time onto radius of sphere, $t \to \log(r)$
- Spatial degrees of freedom are mapped onto spherical shell
- Problem becomes solving a Euclidean QFT on a sphere
- Task:
  - Find best way to discretize sphere (find best "graph")
  - Need to take curvature into account (need some basic gravity theory)
- Goal for this talk:
  - Examine convergence of spectrum of discretized classical field operator to continuum spectrum for different graphs

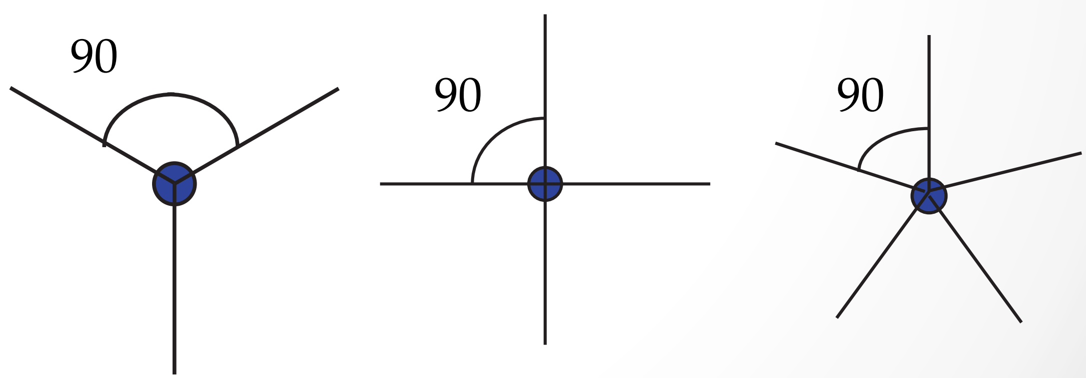
going from continuum to lattice action

- Naïve discretization: $\nabla_\mu \varphi(x) \to \frac{1}{a}(\varphi_{x+a\mu} - \varphi_x)$
  - Not obvious how to do this on a non-toroidal lattice
  - Need to go from sum over μ to a sum over nearest neighbors
  - Nearest neighbor configurations usually vary from site to site

Rectangular 2-torus
all vertices the same

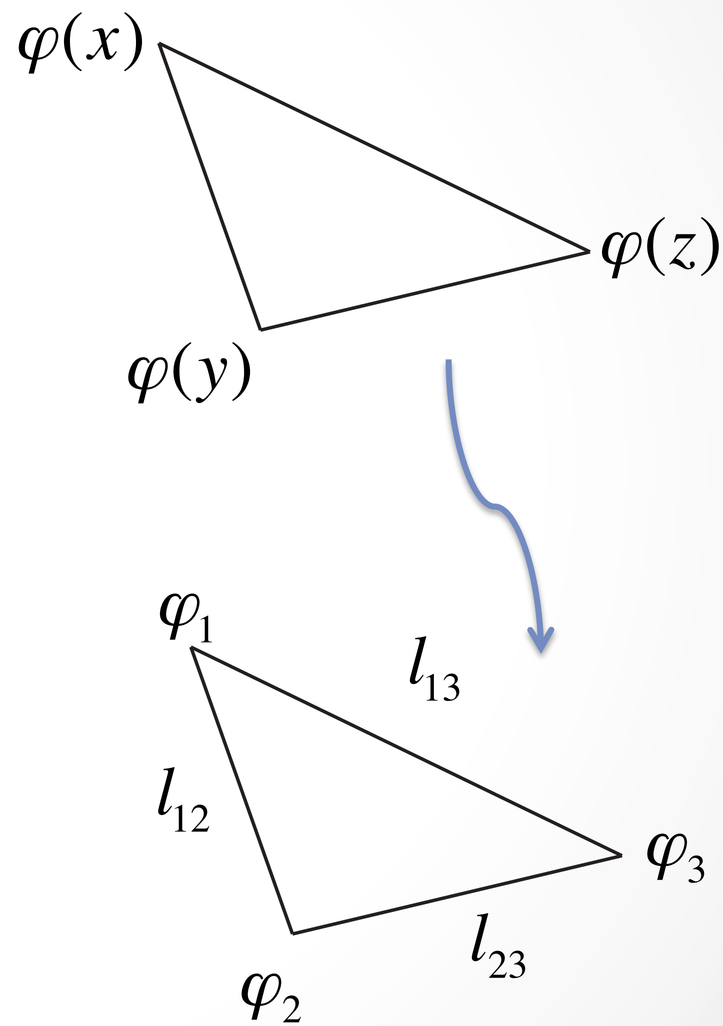


Rectangular 2-sphere
3,4, or 5 neighbors at a vertex



a general approach to discrete curved spaces (2)

- "Regge Calculus"
  - Simplicial gravity
  - Coordinate free
  - All information contained in connectivity of graph and invariant lengths
  - Theory is defined without reference to embedding space
  - The same graph can take on many appearances in embedding space)
    - Basketball can be inflated or flat (caved in)



Regge calc. ctd.

- Task: discretize action of scalar field theory in curved space

$$S = \int d^4x \sqrt{g} \left(g^{\mu\nu} (\partial_\mu \varphi)(\partial_\nu \varphi) + m^2 \varphi^2 \right)$$

- Regge calculus tells us to make the following replacements

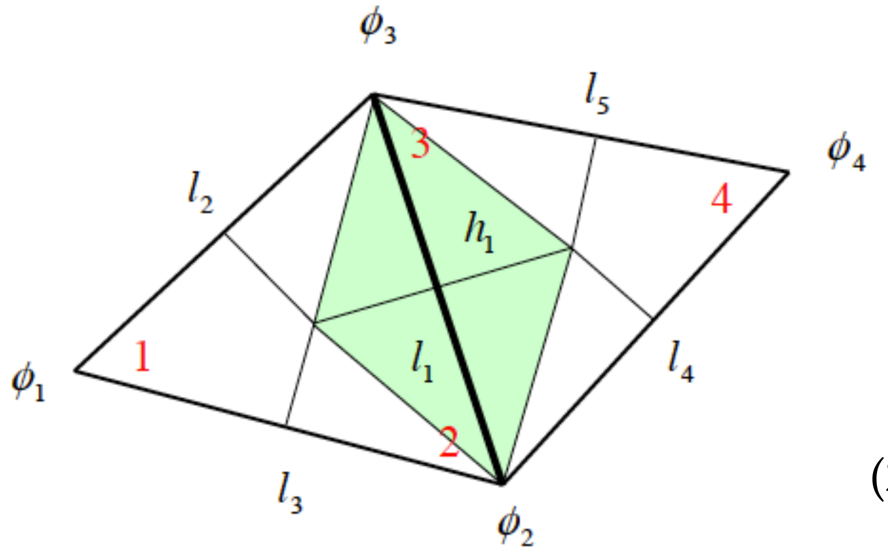
$$g_{\mu\nu} \to g_{ij}(\Delta) = \begin{pmatrix} l_3^2 & (1/2)(-l_1^2 + l_2^2 + l_3^2) \\ (1/2)(-l_1^2 + l_2^2 + l_3^2) & l_2^3 \end{pmatrix}$$

$$\partial_\mu \varphi(x_i) \to \varphi_{i+\mu} - \varphi_i$$

- μ runs over possible directions one can move away from a vertex within a given triangle

Regge calc. ctd.

- Arrive at lattice action: $S = \frac{1}{2}\sum_{<ij>} A_{ij}\left(\frac{\phi_i - \phi_j}{l_{ij}}\right)^2$
- A_{ij} can be expressed in terms of invariant lengths
  - "Voronoi dual area"
  - Usually varies from link to link except in special cases
  - Think of as "weight" of link



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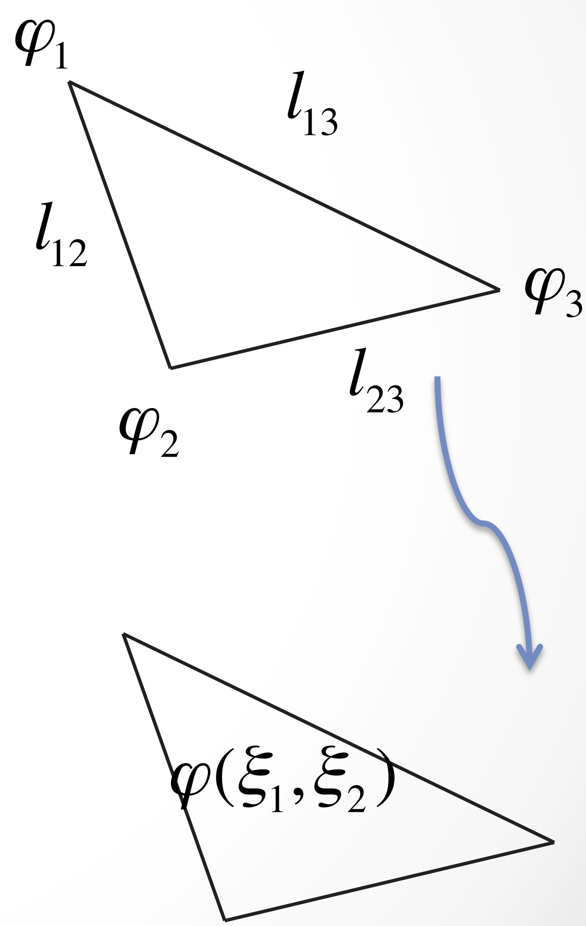
A note on FEM (3)

- If one defines a linear interpolation of the field over each triangle and uses it to compute the finite element action, one arrives at the same answer as Regge calculus
  - Again, each link is weighted by the corresponding Vononoi dual area
  - Regge calculus and linear FEM are the same thing



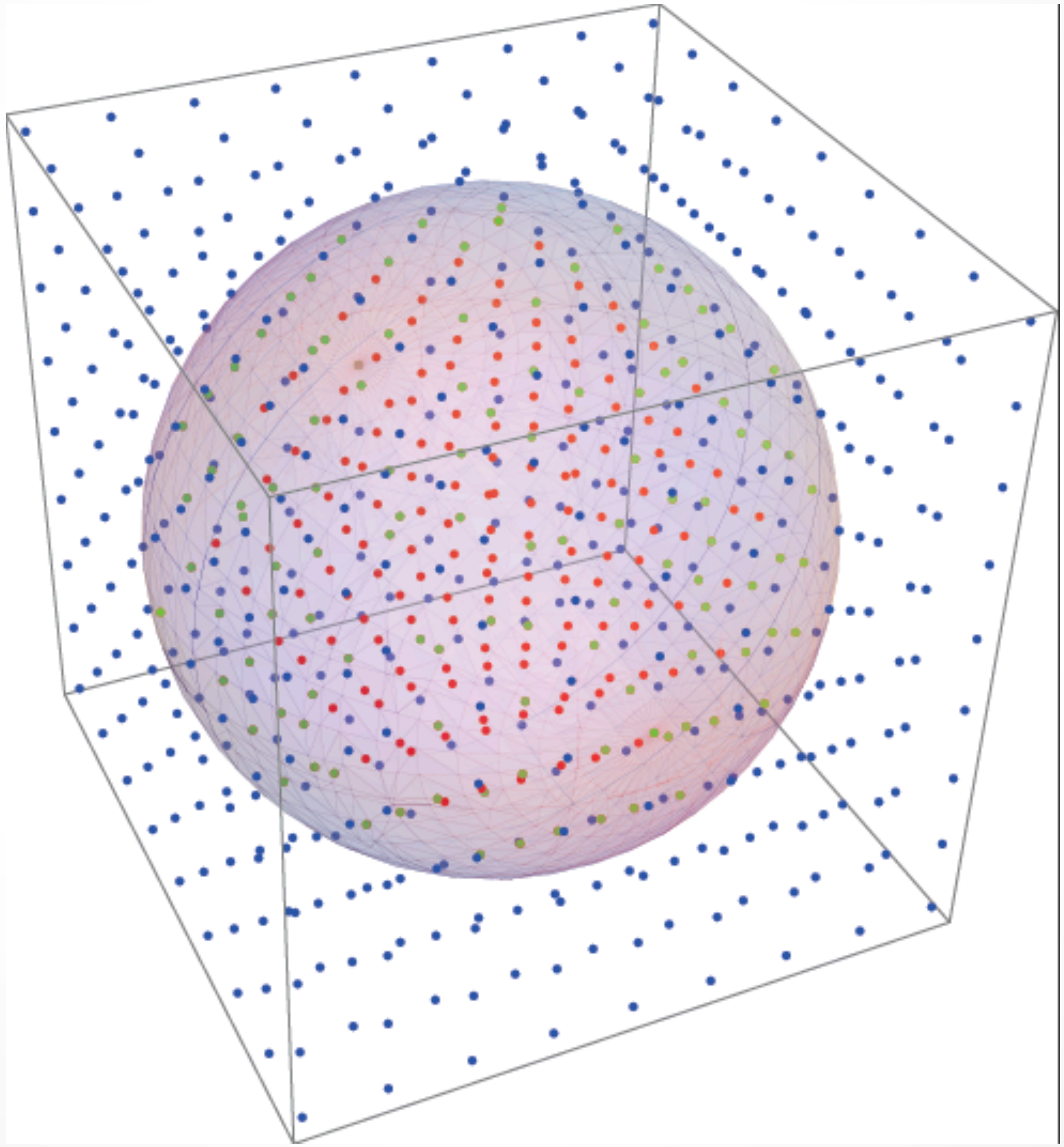
Spherical lattices

- Refined cube
  - Construction
  - Pros/Cons
  - Construction of discrete field op
- Refined Icosahedron
  - Construction
  - Pros/Cons
  - Constructions of discrete field op

Constructing a refined cubic mesh

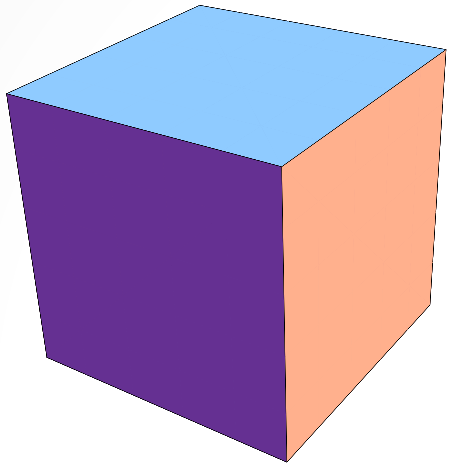
- Draw a cubic grid of size (s+2)^3
  - Edges always get cut off, so the final object has side length s
  - Call s the "refinement" so s=1 is just a cube.
- Draw an inscribed sphere
- Throw out all points outside the spherical volume
- Of the points remaining, figure out which ones are on the surface
  - For each small cube, check if its neighbor cube is completely contained in the spherical volume. If not, then the face of the small cube in the direction of the missing neighbor cube is on the surface

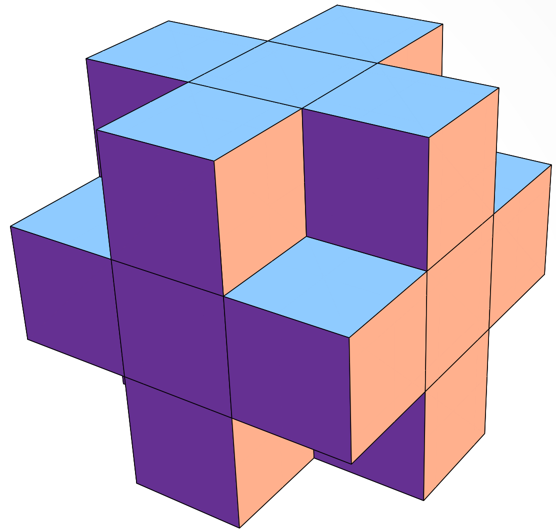
Refined cubic meshes



Refined cubic meshes

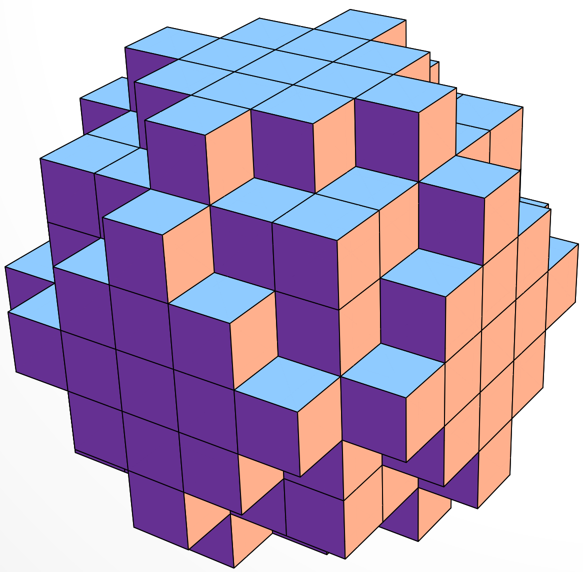
S=1

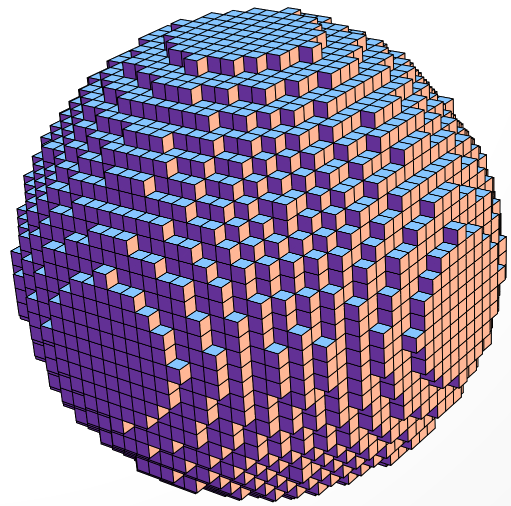




S=3

S=7

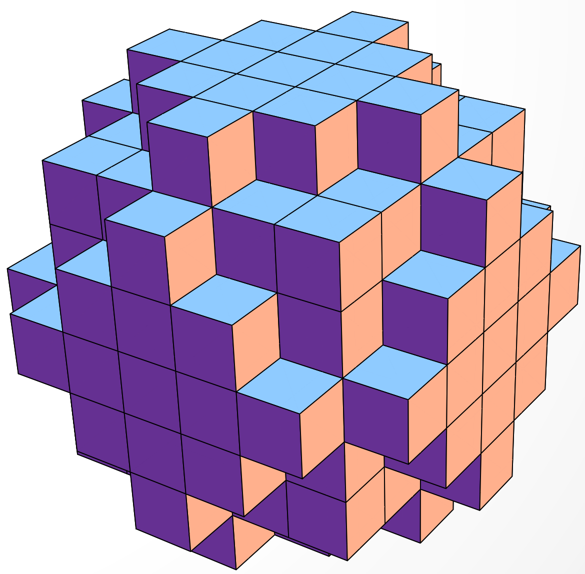




S=31

Refined cubic meshes: pros/cons

- Pros
  - All size lengths are equal
  - All faces are the same (squares)
  - Voronoi dual areas are all equal
    - Regge calculus action simplifies greatly (next slide)
  - Curvature at a vertex determined entirely by number of neighbors
  - We know the embedding space, so it's easy to define a global coordinate system for spinor/vector fields (future work)
- Cons
  - Vertices do not lie on sphere
  - Seems to converge more slowly to continuum (more on this soon)



Note on Laplace operator on refined cube

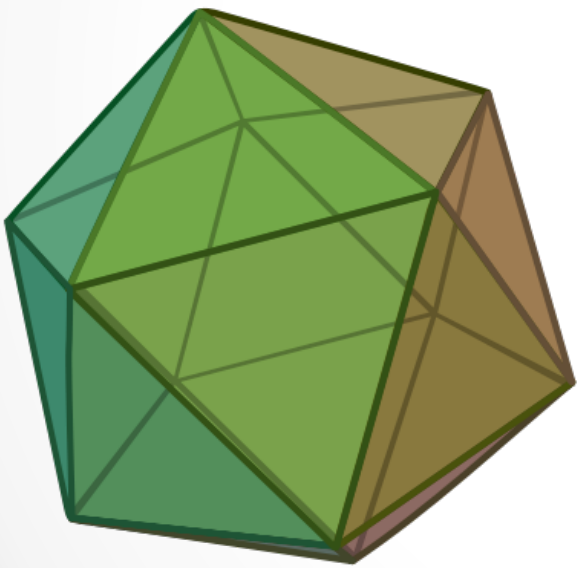
- Voronoi areas and edge lengths are the same for all links, so the Regge calc action becomes

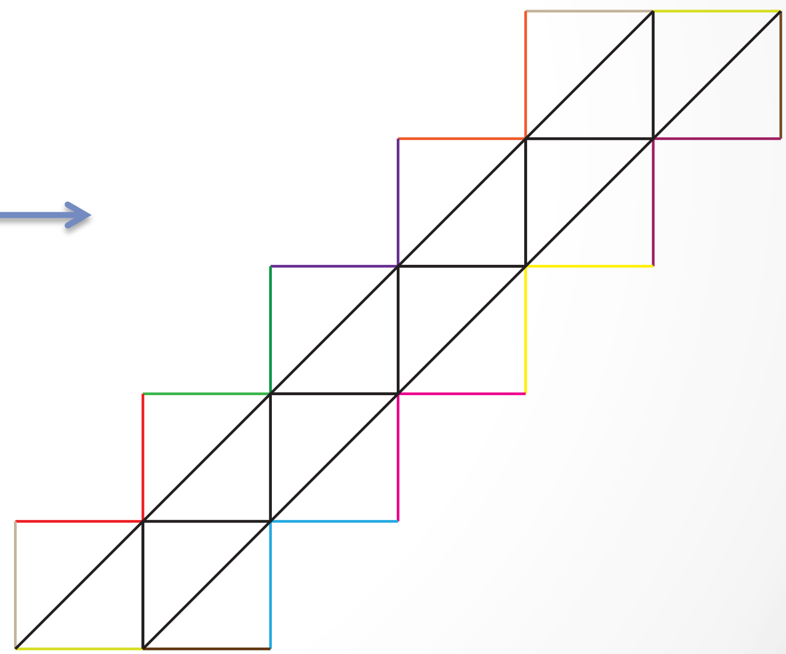
$$S = \frac{1}{2}\sum_{<ij>} A_{ij}\left(\frac{\phi_i - \phi_j}{l_{ij}}\right)^2 = c\sum_{<ij>}\left(\phi_i - \phi_j\right)^2 = c\sum_i \sum_{\pm\mu}\left(\phi_{i+\mu} - \phi_i\right)^2$$

- This is the same as the naively discretized action

Constructing refined icosahedron (4)

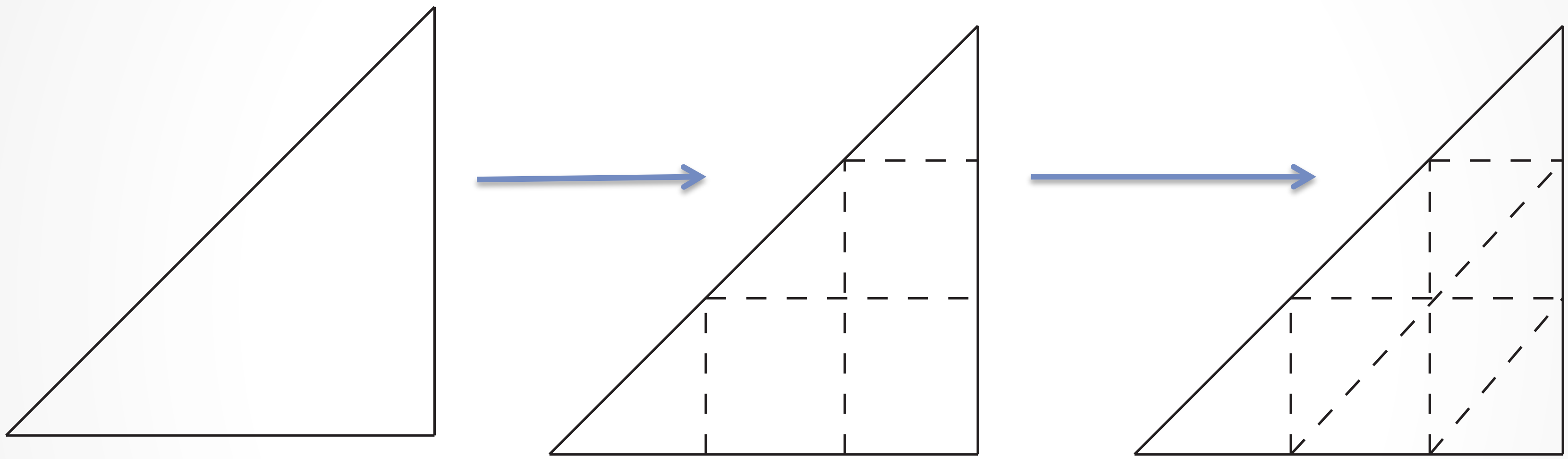
- Easy to visualize by mapping onto a flat graph
  - The flat graph just gives the connectivity
    - Notice, on icosahedron triangles are equilateral and on flat graph they are isosceles. Flat graph is warped.
  - Edges of same color are the same edge on the icosahedron





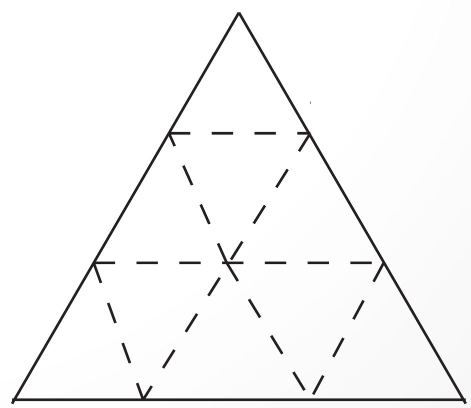
Constructing refined icosahedron

- To refine the graph, divide up each triangle as shown below (refinement s=3)



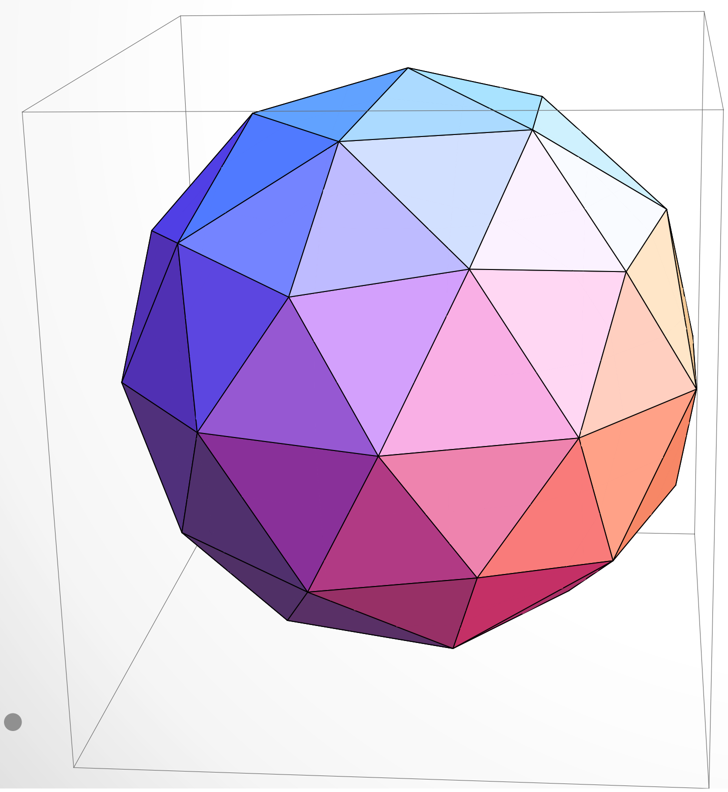
- On icosahedron

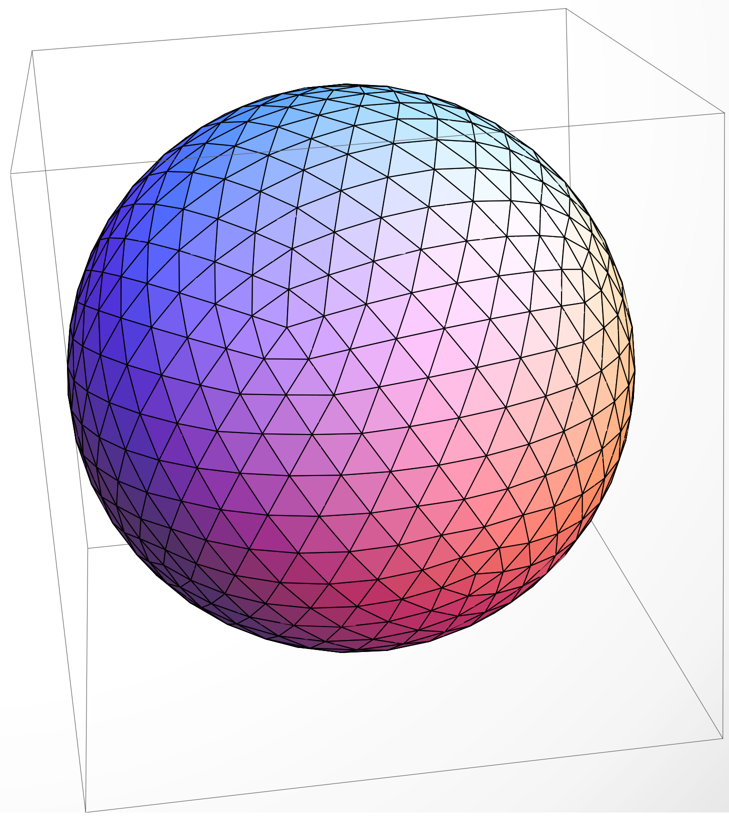
(equilateral triangles), looks like:



Constructing refined icosahedron

- So far we've just made a more highly discretized icosahedron. It is not yet spherical.
- To make it approach the sphere, stereographically project points onto sphere





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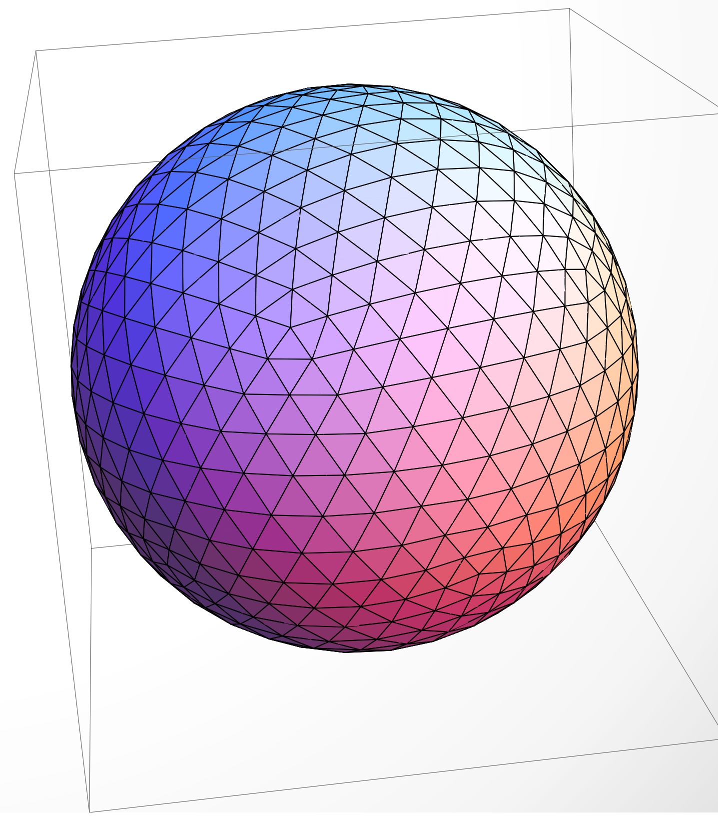
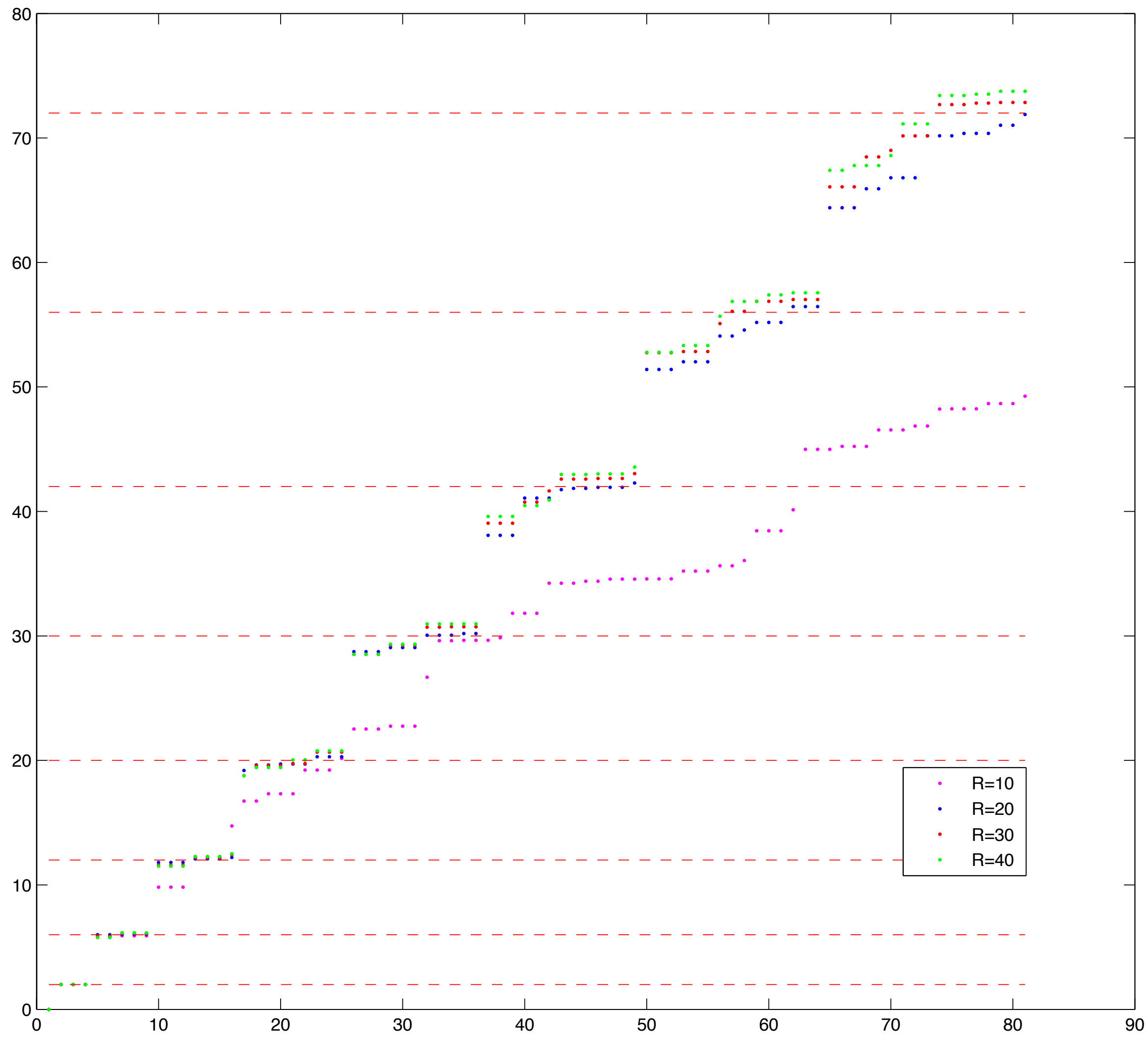
Refined icosahedron: pros/cons

- Pros
  - Largest discrete rotation group built in
  - Triangular faces → simlicial mesh
    - Easy to apply Regge calc
  - Can construct without referring to embedding space
- Cons
  - Edge lengths not equal
  - Shapes of faces vary
  - Voronoi (dual) areas are not equal

$$S = \frac{1}{2}\sum_{<ij>} A_{ij}\left(\frac{\phi_i - \phi_j}{l_{ij}}\right)^2$$



Convergence of discrete laplacian spectrum



Convergence of discrete laplacian spectrum

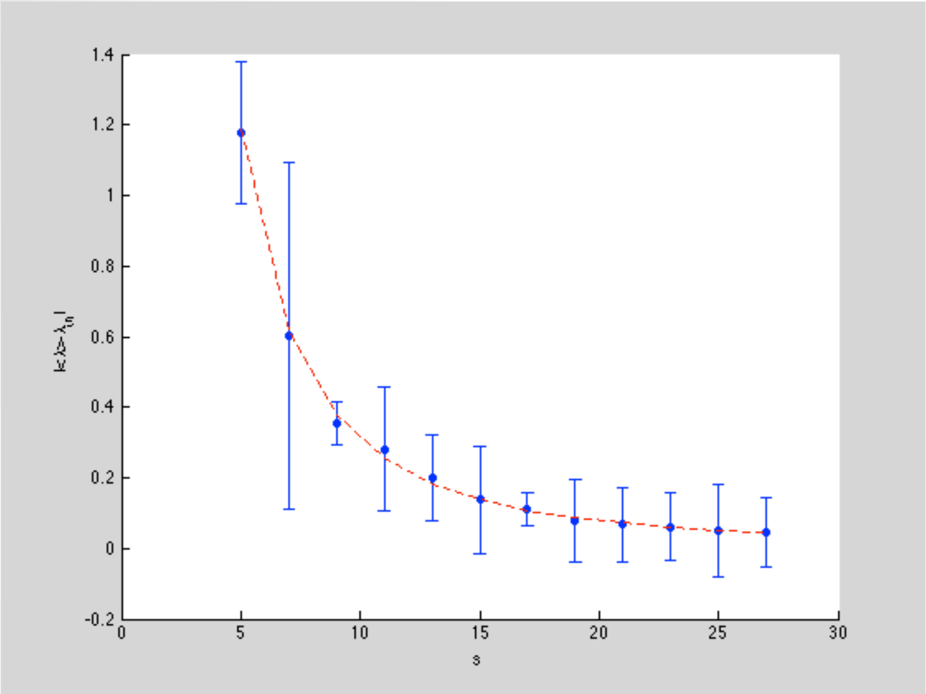
- Convergence of the mean at each level L. Expect:

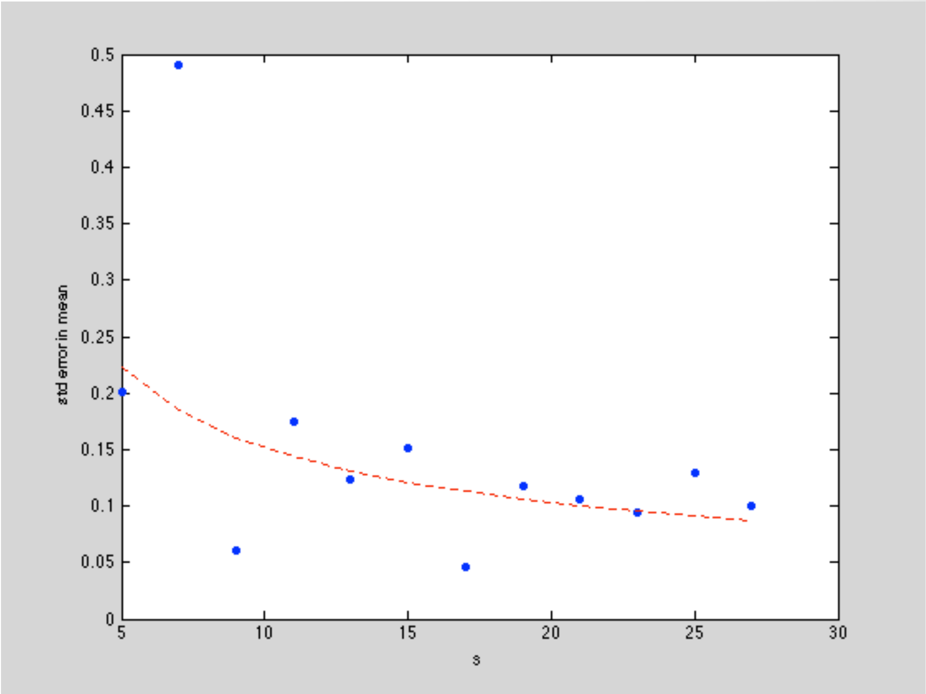
$$\langle \lambda \rangle = \lambda_{th} + \frac{a(\lambda)}{s^2} \qquad \lambda_{th} = L(L+1)$$

- Convergence of standard error in the mean

$$\delta_\lambda = \frac{b(\lambda)}{s^2}$$

Refined cube. L=3

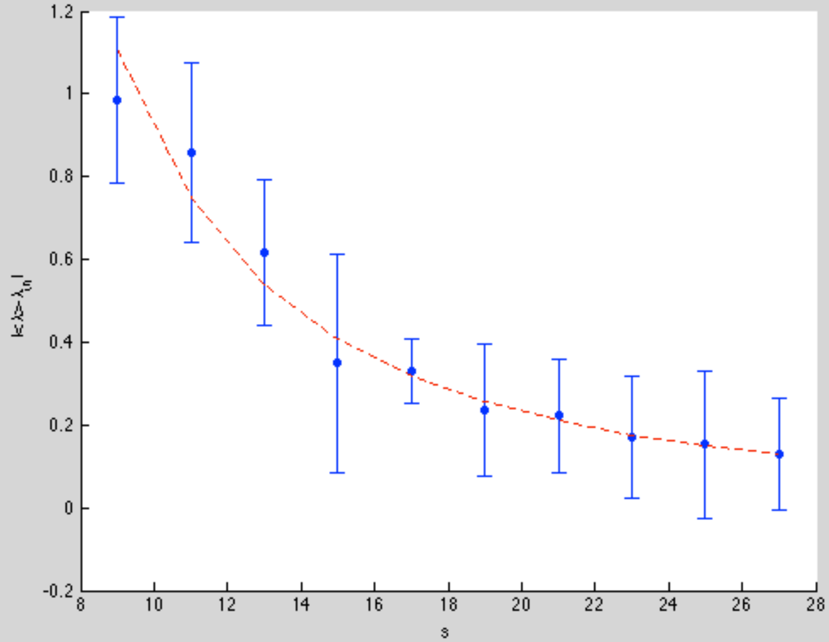


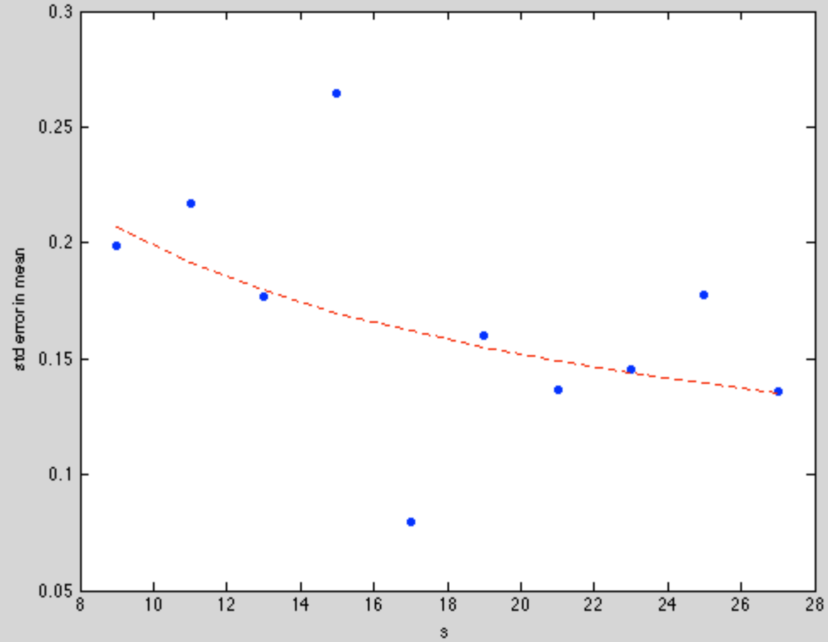


$n = 2.06$

$n = 0.56$

Refined cube. L=4





$n = 1.95$

$n = 0.38$

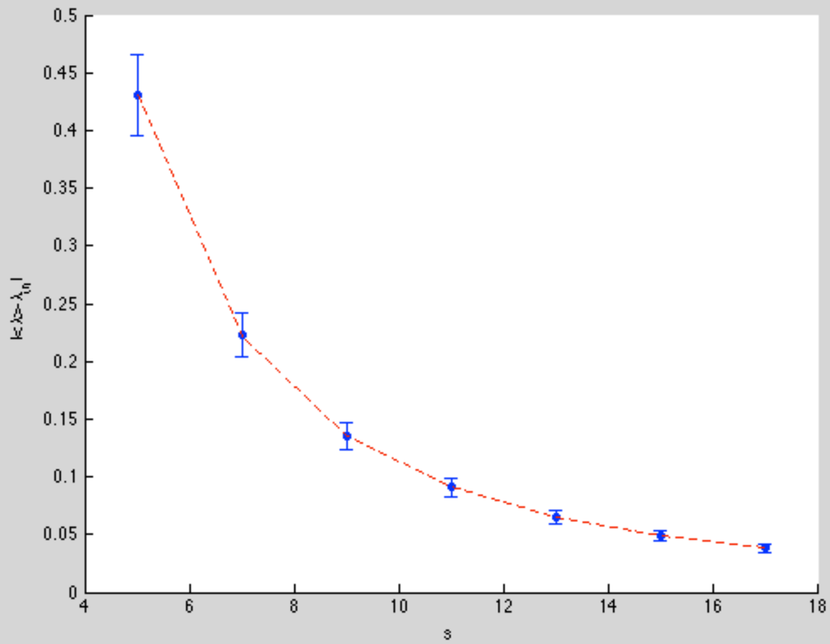
Why aren't the standard errors falling off like 1/s^2?

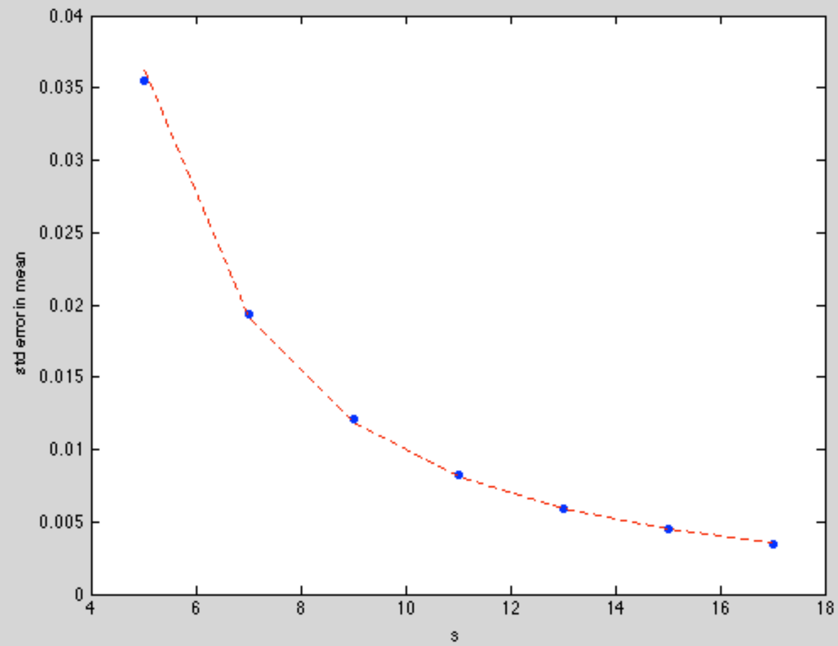
- Voronoi dual areas for LINKS are equal, but the SITES still "own" uneven amounts of the surface
- Need to solve generalized eigenvalue problem

$$O_{xy}\varphi_y = w_y\varphi_y$$

- Options for vertex weights
  - Voronoi dual area
  - "1/3 area rule"

Refined icosahedron. L=3

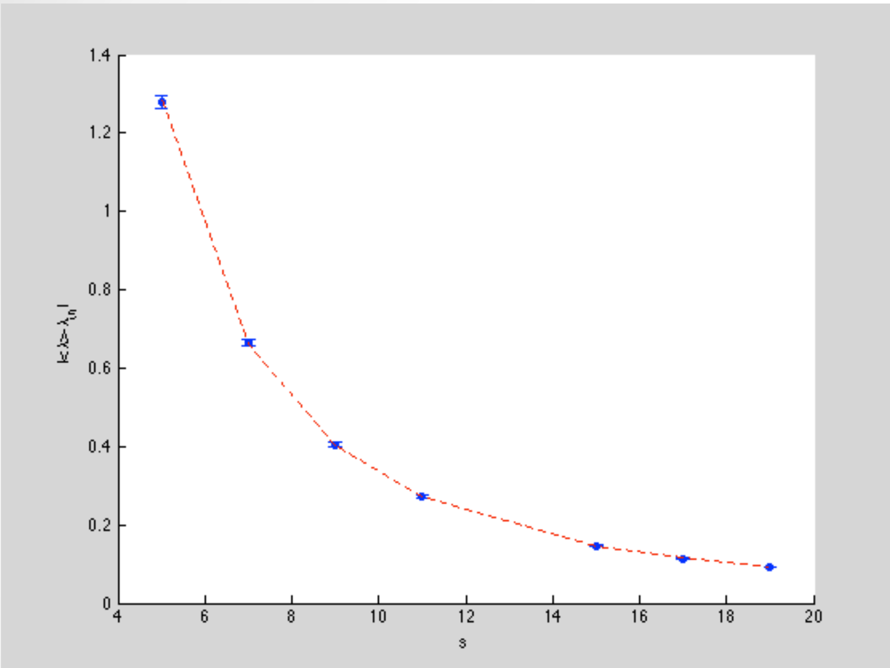




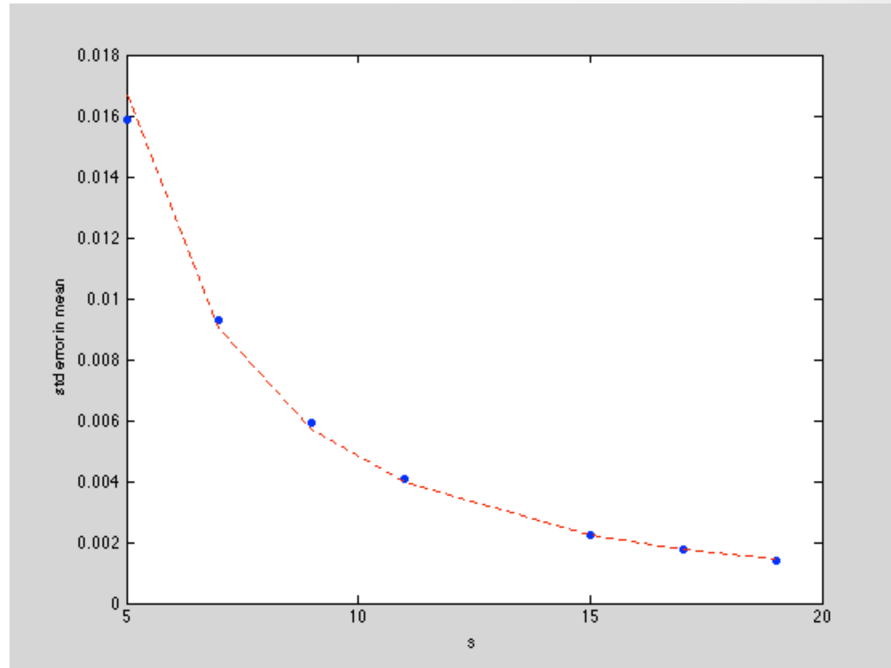
$n = 1.98$

$n = 1.90$

Refined icosahedron. L=4

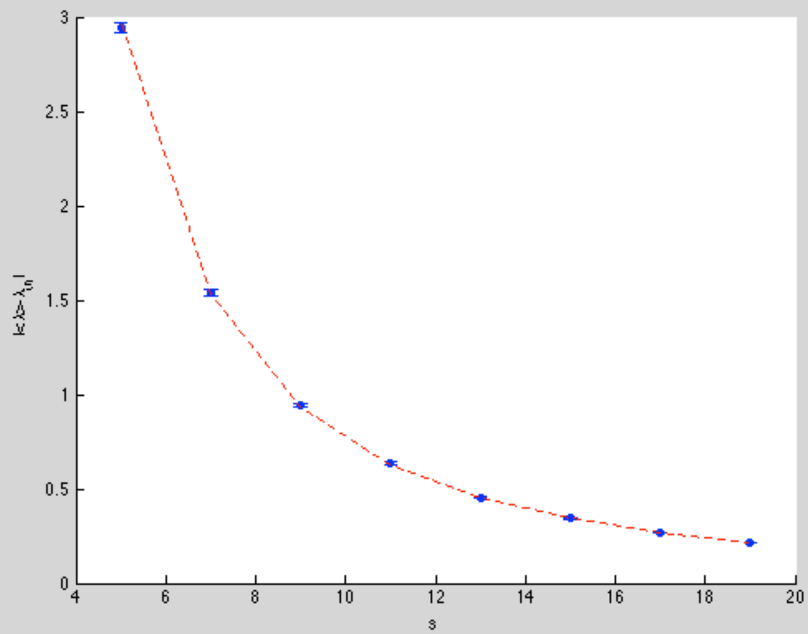


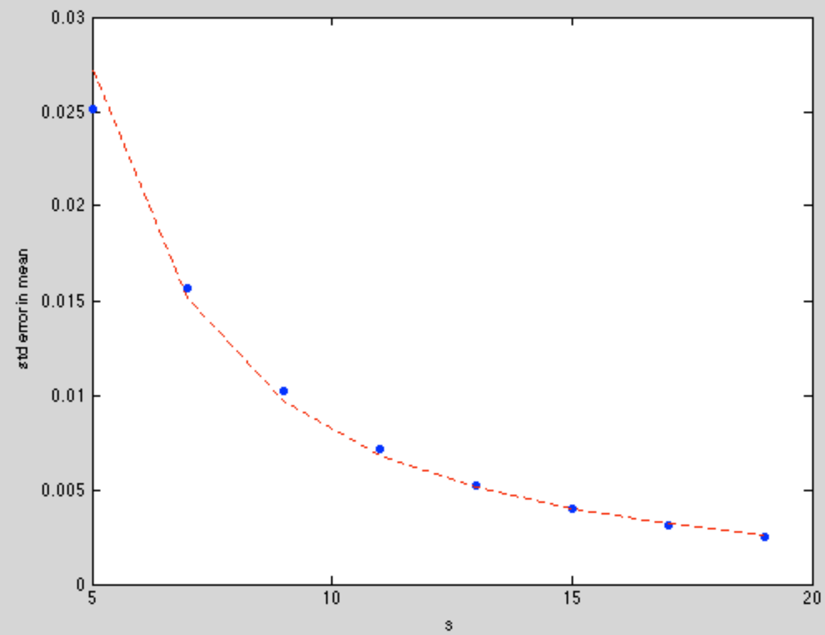
$n = 1.98$



$n = 1.83$

Refined icosahedron. L=5

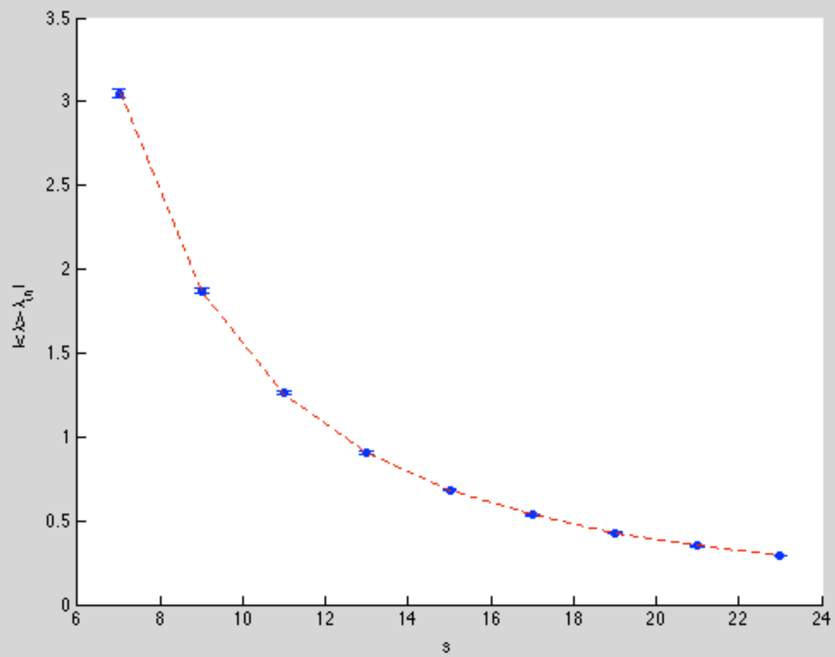


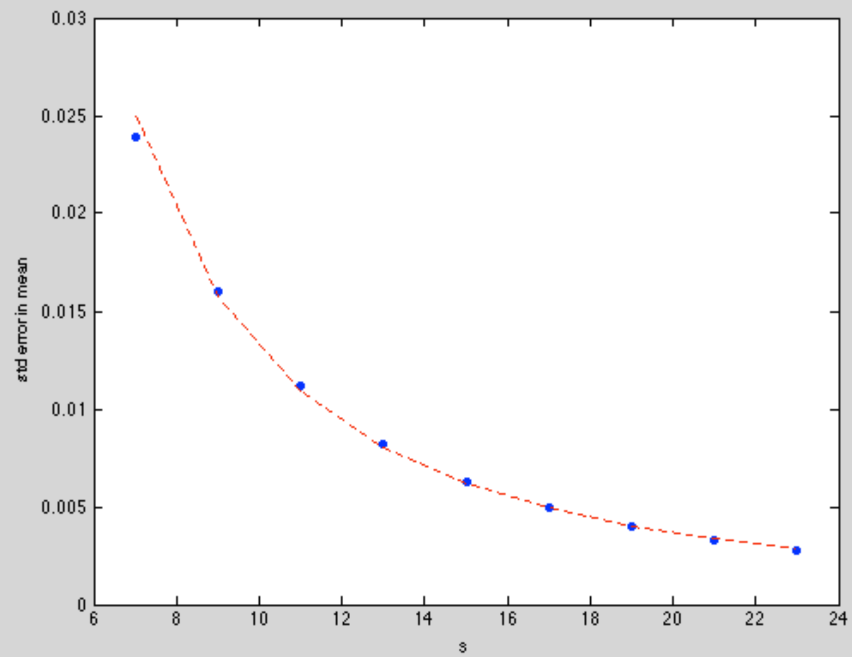


$n = 1.97$

$n = 1.75$

Refined icosahedron. L=6





$n = 1.97$

$n = 1.83$

Sources

1. http://www.philohome.com/sphere/sphere.htm
2. Hamber, Herbert. "Discrete and Continuum Quantum Gravity." arXiv: 0704.2895
3. Brower, Richard C., Michael Cheng, and George T. Fleming. "Improved Lattice Radial Quantization." (2013).
4. Brower, Richard C., George T. Fleming, and Herbert Neuberger. "Radial Quantization for Conformal Field Theories on the Lattice." *arXiv preprint arXiv: 1212.1757* (2012).