Meson spectral function: an application of the Maximum Entropy Method

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- Meson correlations on lattice and the spectral functions
- Maximum Entropy Method in lattice QCD
- Numerical results : charmonium spectral functions at zero and finite temperature
- Conclusions and Outlook

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Meson correlators and spectral functions

Lattice QCD is formulated in imaginary time

$$G(\tau,\vec{p},T)=\int d^3x\, e^{i\vec{p}\cdot\vec{x}}\left\langle J_H(\tau,\vec{x})\, J_H^{+}(0,0)\right\rangle,$$

$$J_H(\tau,\vec{x})=\overline{\psi}(\tau,\vec{x})\Gamma_H\psi(\tau,\vec{x})$$

$$\Gamma_H=1,\ \gamma_5,\ \gamma_\mu,\ \gamma_5\cdot\gamma_\mu$$

Physical processes take place in real time

$$D^{>}(t,\vec{p},T)=\int d^3x\, e^{i\vec{p}\cdot\vec{x}}\left\langle J_H(t,\vec{x})\, J_H\ (0,0)\right\rangle,$$

$$D^{<}(t,\vec{p},T)=\int d^3x\, e^{i\vec{p}\cdot\vec{x}}\left\langle J_H(0,\vec{0})\, J_H\ (t,\vec{x})\right\rangle$$

$$\frac{D^{>}(\omega)-D^{<}(\omega)}{2\pi}=\frac{1}{\pi}\mathrm{Im}\, D_R(\omega)=\sigma(\omega)$$

$$\boxed{G(\tau,T)=D^{>}(-i\tau)}$$

$$\Downarrow$$

$$\boxed{G(\tau,T)=\int_0^\infty d\omega\,\sigma(\omega,T)\frac{\cosh(\omega(\tau-1/(2T)))}{\sinh(\omega/(2T))}}$$

if $T=0$ and $\sigma(\omega)=\sum_n A_n\delta(\omega-E_n)$ $\Rightarrow$ $G(\tau)=A_0e^{-E_0\tau}+A_1e^{-E_1\tau}+\ldots$

fit the large distance behavior of the lattice correlation functions

This is not possible for $T>0$, $\tau_{max}=1/T$
and in the case of resonances, e.g. $R(\omega)=\frac{\sigma_{e^+e^-\to hadrons}}{\sigma_{e^+e^-\to\mu^+\mu^-}}=\sigma(\omega)/\omega^2$

Reconstruction of the spectral functions : MEM

$$G(\tau,T)=\int_0^\infty d\omega\sigma(\omega,T)\cdot K(\omega,T)$$

$\mathcal{O}(10)$ data and $\mathcal{O}(100)$ degrees of freedom to reconstruct

Bayesian techniques: find $\sigma(\omega,T)$ which maximizes

$$P[\sigma|DH]\sim P[D|\sigma H]P[\sigma|H]$$ (Bayes' theorem)

data

Prior knowledge

prior probability

$H:\ \sigma(\omega,T)>0$ ⇒ Maximum Entropy Method (MEM): $P[\sigma|H]=e^{\alpha S}$

Asakawa, Hatsuda, Nakahara, PRD 60 (99) 091503, Prog. Part. Nucl. Phys. 46 (01) 459

$$P[\sigma|DH]=P[\sigma|D\alpha m]=\exp(-\frac{1}{2}\chi^2+\alpha S)$$

Likelihood function

Shannon-Janes entropy:

$$S=\int_0^\infty d\omega[\sigma(\omega)-m(\omega)-\sigma(\omega)\ln\frac{\sigma(\omega)}{m(\omega)}]$$

$m(\omega)$ - default model $\quad m(\omega\gg\Lambda_{QCD})=m_0\omega^2$ -perturbation theory

Procedure for calculating the spectral functions

How to find numerically a global maximum in the parameters space of *O*(100) dimenisons for fixed α ?

$$\sigma(\omega) = m(\omega)\exp\left[\sum_{i=1}^{N} s_i u_i(\omega)\right], \quad N \le N_\tau/2$$

Find the basis $u_i(\omega)$ through

SVD of $K = U\Sigma V$, $u_i(\omega_j) = U_{ji}$ Bryan, Europ. Biophys. J. 18 (1990) 165

or use $u_i(\omega) = K(\omega, \tau_i)$ Jakovác, P.P. , Petrov, Velytsky, PRD 75 (2007) 014506

maximization of $P[\sigma|D\alpha m]$ reduces to minimization of

$$U = \frac{\alpha}{2}\sum_{i,j=1}^{N} s_i C_{ij} s_j + \sum_{l=0}^{N_\omega}\sigma(\omega_l)\Delta\omega - \sum_{i=1}^{N}\bar{G}(\tau_i) s_i$$

covariance matrix ensemble average

which can be done using Levenberg-Marquardt algorithm $\Rightarrow$ $\hat{\sigma}_\alpha$

Procedure for calculating the spectral functions (cont'd)

How to deal with the α-dependence of the result ?

$$\sigma(\omega) = \int d\alpha \; \hat{\sigma}_\alpha(\omega) \;\; P[\alpha|Dm]$$

For good data $P[\sigma|D\alpha m]$ is sharply peak around $\sigma(\omega) = \hat{\sigma}_\alpha(\omega)$ and using Bayes' theorem

$$\begin{aligned}
P[\alpha|Dm] &\sim \int [d\sigma] P[D|\sigma\alpha m] P[\sigma|\alpha m] P[\alpha|m] \\
&\sim P[\alpha|m] \int [d\sigma] \exp\left[-\frac{1}{2}\chi^2 + \alpha S\right] \\
&\sim P[\alpha|m] \exp\left[\frac{1}{2}\sum_k \frac{\alpha}{\alpha+\lambda_k} + \alpha S(\hat{\sigma}_\alpha) - \frac{1}{2}\chi^2(\hat{\sigma}_\alpha)\right]
\end{aligned}$$

λ_k are the eigenvalues of $\Lambda_{ll'} = \frac{1}{2}\sqrt{\sigma_l}\frac{\partial \chi^2}{\partial\sigma_l \partial\sigma_{l'}}\sqrt{\sigma_{l'}}|_{\sigma=\hat{\sigma}_\alpha}$ and common choices for $P[\alpha|m]$ are $P[\alpha|m] = const$ and $P[\alpha|m] = 1/\alpha$.

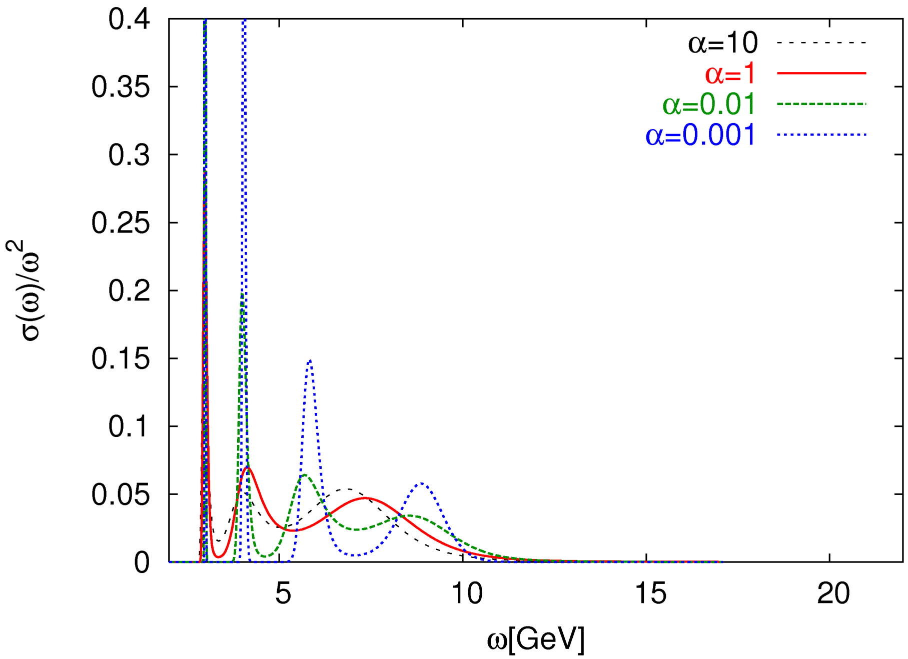
In practice $P[\alpha|Dm]$ is peaked at some α_{max}.

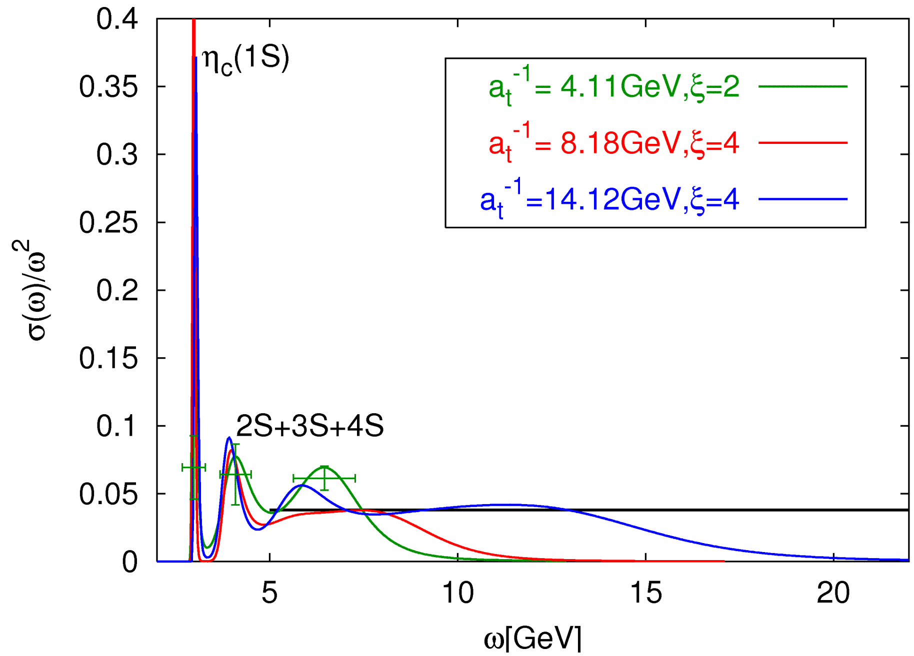
Charmonium spectral functions at T=0

Anisotropic lattices: $16^3 \times 64, \xi = 2$ $16^3 \times 96, \xi = 4$, $24^3 \times 160, \xi = 4$
$L_s = 1.35 - 1.54$fm, #configs=500-930;
Wilson gauge action and Fermilab heavy quark action

Pseudo-scalar (PS) $\rightarrow$ S-states

Jakovác, P.P. , Petrov, Velytsky, PRD 75 (2007) 014506

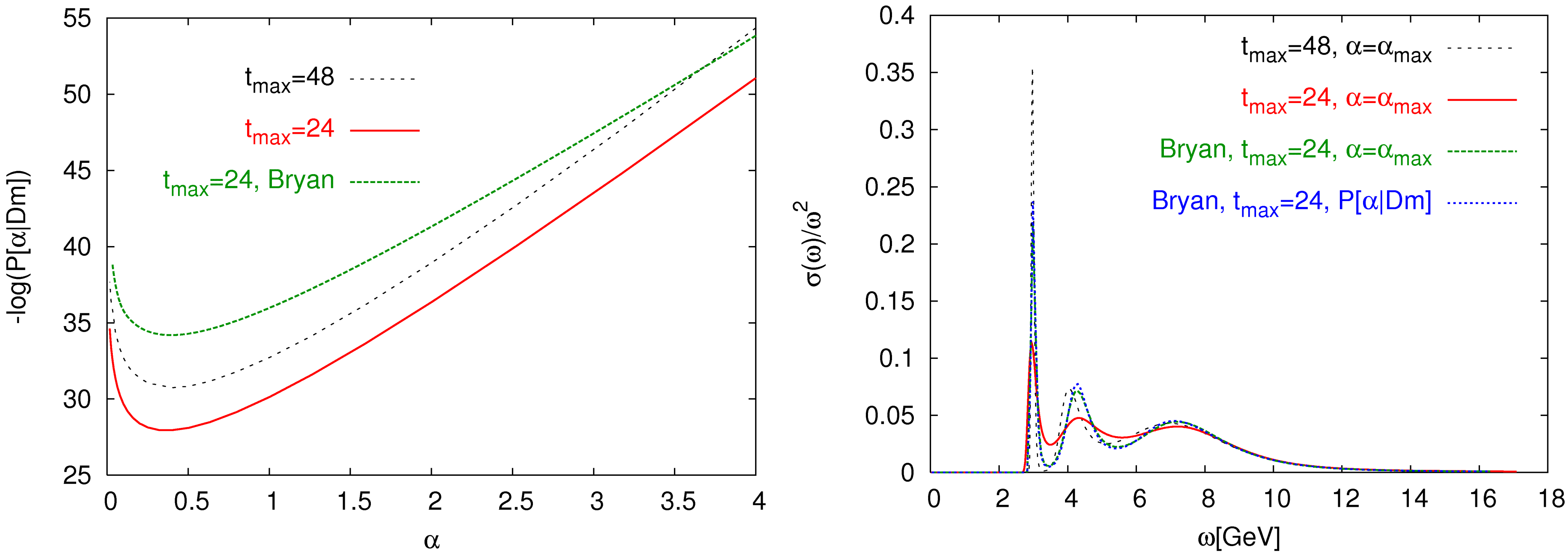




For $\omega > 5$ GeV the spectral function is sensitive to lattice cut-off ;
good agreement with 2-exponential fit for peak position and amplitude

Charmonium spectral functions at T=0 (cont'd)

PS, $16^3 \times 96$, $a_t^{-1} = 8.18$ GeV, $\xi = 4$



$P[\alpha|Dm]$ has a well defined maximum at some α_{max}

Bryan algorithm and the new algorithm give similar results, but the use of the former is limited $\tau_{max} = 24$.

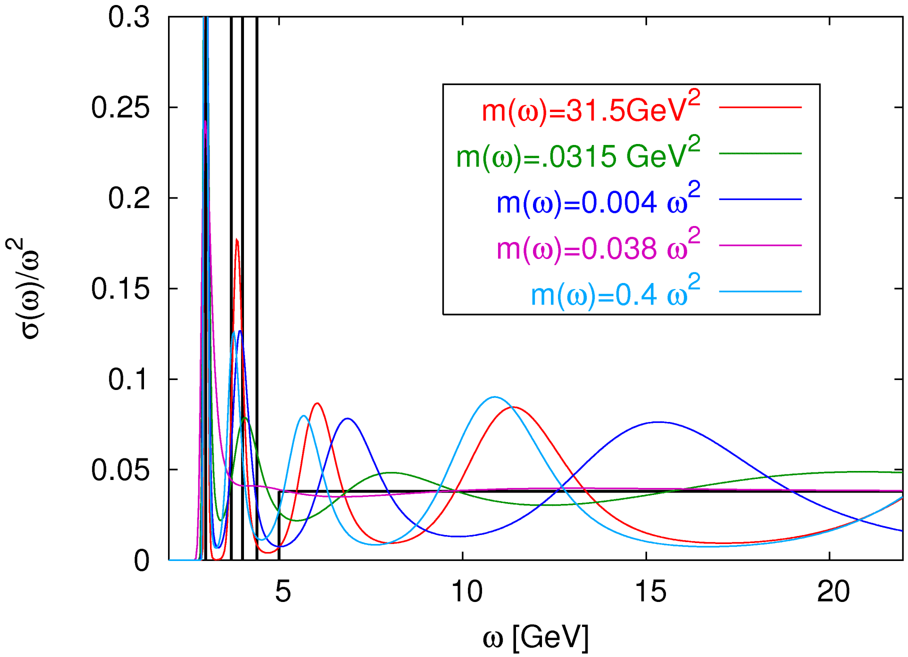
Charmonium spectral functions at T=0 (cont'd)

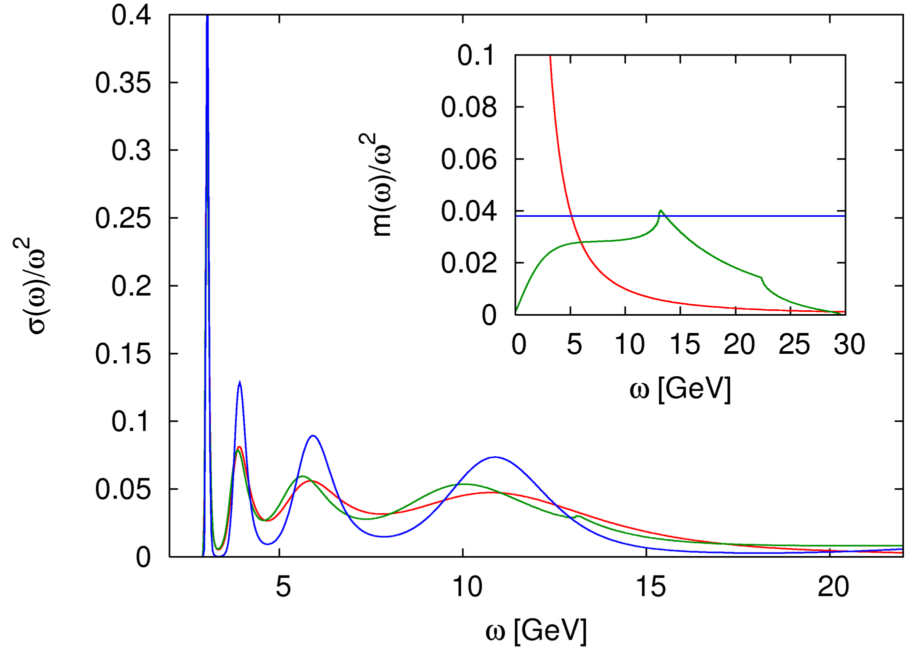
What states can be resolved and what is the dependence on the default model ?

Reconstruction of an input spectral function :

Lattice data in PS channel for:

$$a_t^{-1} = 14.12\mathrm{GeV},\ N_t = 160$$





Ground states is well resolved, no default model dependence;
Excited states are not resolved individually, moderate dependence on the default model;
Strong default model dependence in the continuum region, $\omega > 5$ GeV

Comparison of MEM and 2-exponential fit

Meson masses , $\beta = 6.5,\ \xi = 4, 24^3 \times 160$:

	MEM	2-exp fit
$m_{ps}(n=1)$	0.2147(7)	0.2154(2)
$m_{ps}(n=2)$	0.281(8)	0.285(2)
$m_{sc}(n=1)$	0.255(5)	0.251(1)[5]

Amplitudes, $A = \int_{peak} d\omega \sigma(\omega)$, $\beta = 6.5,\ \xi = 4, 24^3 \times 160$:

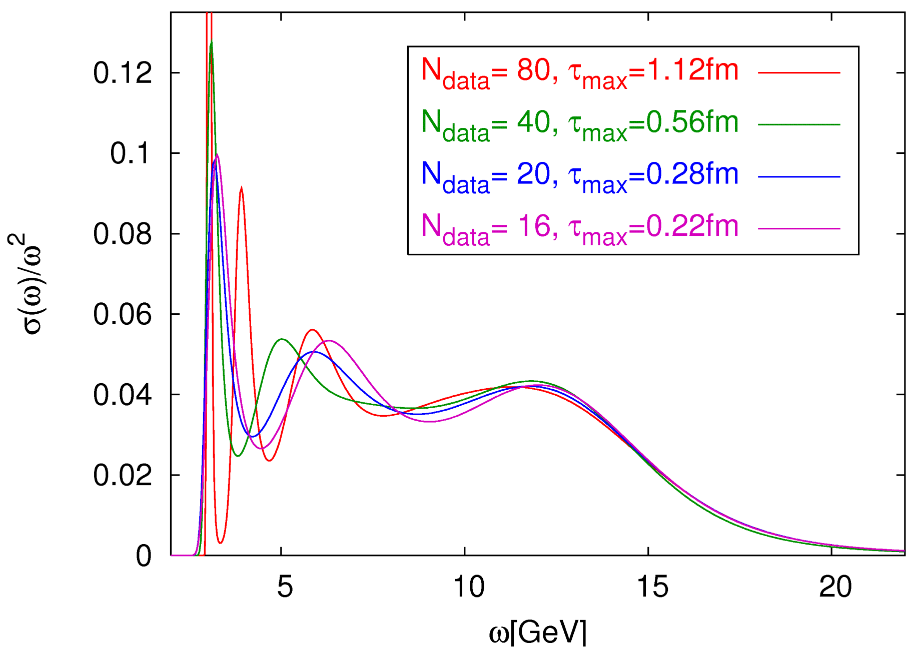
	MEM	2-exp fit
$A_{ps}(n=1)$	0.042(2)	0.0432(5)
$A_{ps}(n=2)$	0.107(13)	0.117(7)
$A_{sc}(n=1)$	0.028(4)	0.021(1)[8]

Charmonia spectral functions at T>0

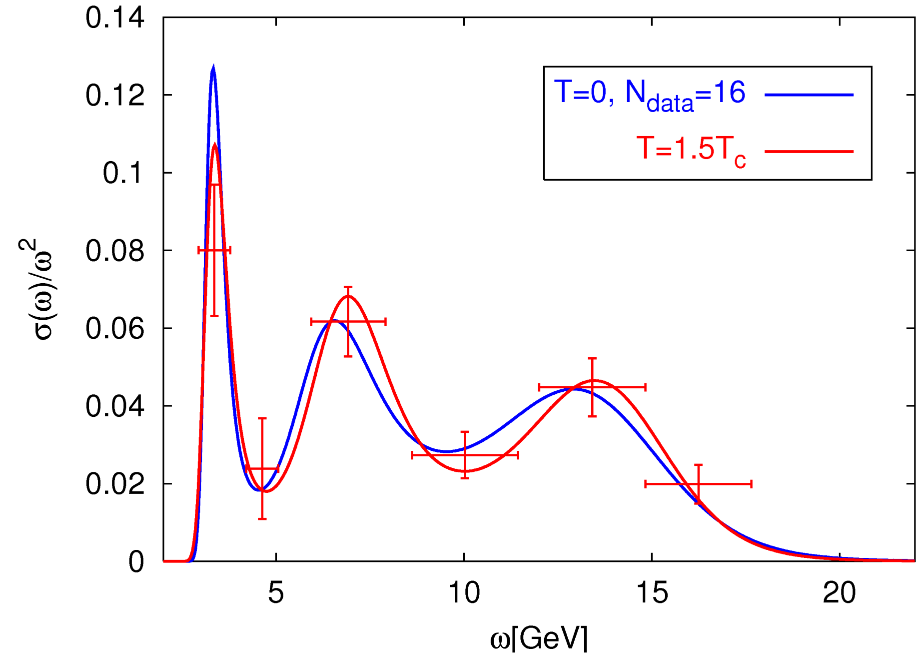
$$T = 1/(N_t a) \quad \leftrightarrow \quad \tau_{max} = 1/(2T)$$

$$\text{PS, } 24^3 \times N_t,\ a_t^{-1} = 14.12 \text{ GeV},\ \xi = 4$$

$T = 0, \quad N_t = 160$



$T = 1.5 T_c, \quad N_t = 32$



ground state peak is shifted, excited states are not resolved when $\tau_{max}, \quad N_{data}$ become small

no temperature dependence in the PS spectral functions within errors

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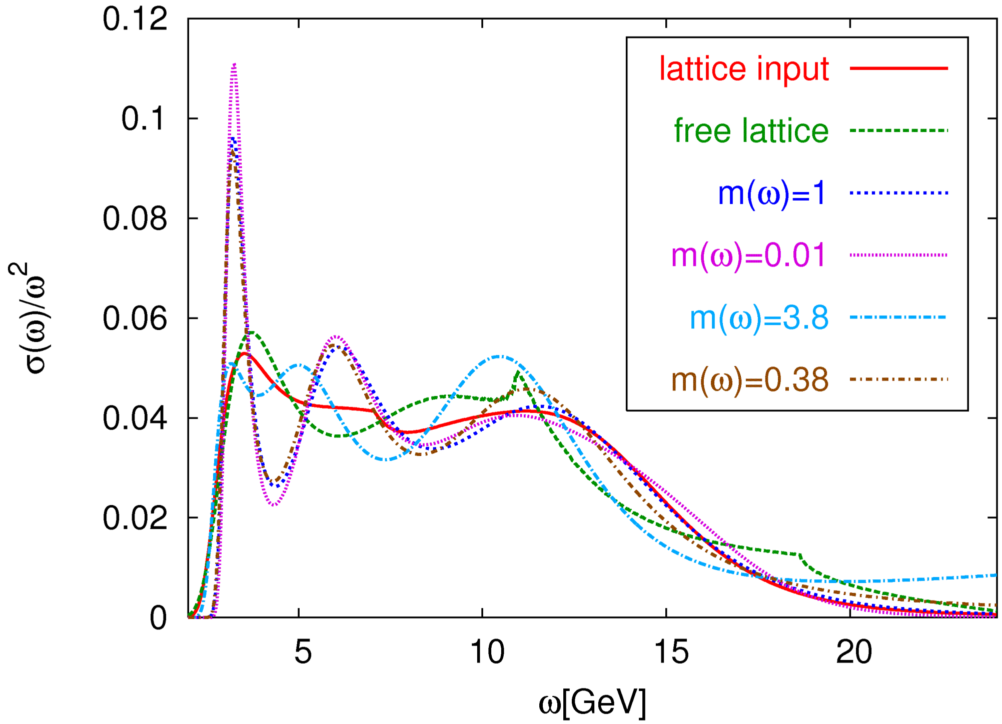
Charmonia spectral functions at T>0 (cont'd)

PS, $24^3 \times 40$, $a_t^{-1} = 14.12$ GeV, $\xi = 4, T = 1.2T_c$



there is a strong dependence on the default model $m(\omega)$ at finite temperature

Conclusions

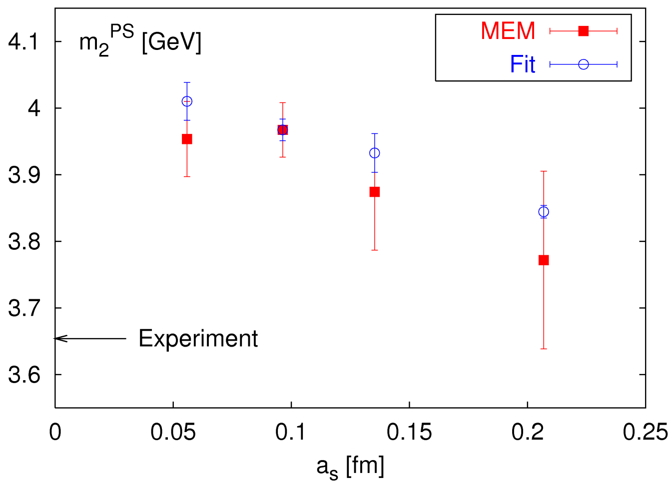
- MEM offers an alternative possibility to analyze meson correlation functions, and we have seen that the meson spectral function extracted from lattice QCD have the expected form (resonances + continuum)

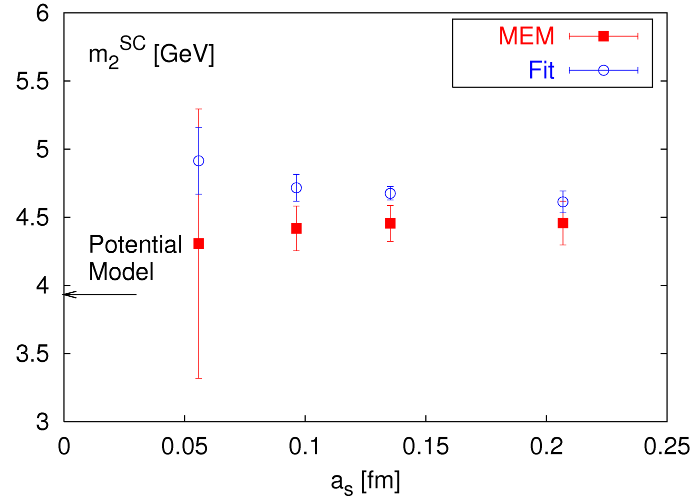
- Results obtained from MEM are compatible with multi-exponential fit, however, for charmonium correlators there is no real advantage of using MEM over the conventional method.

- Extracting width of resonances may be difficult through MEM

- Reliable calculation of meson properties at finite temperature using MEM is difficult with the lattice data available at present

What is the physics behind the 2nd and higher peaks ??

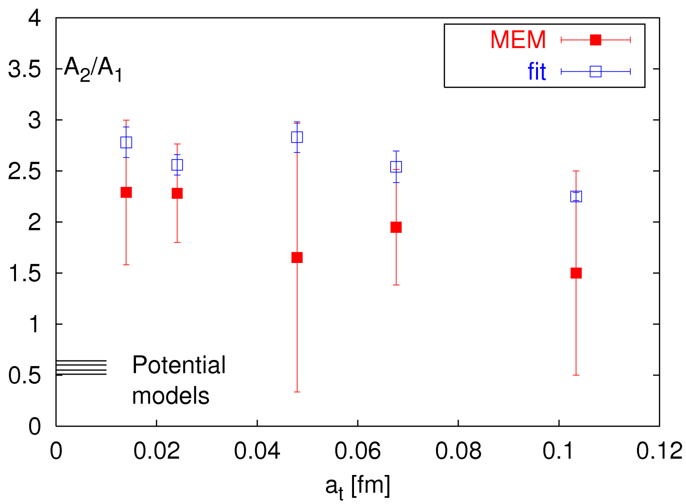




m_2 does not approach the expected value as $a_s \to 0$

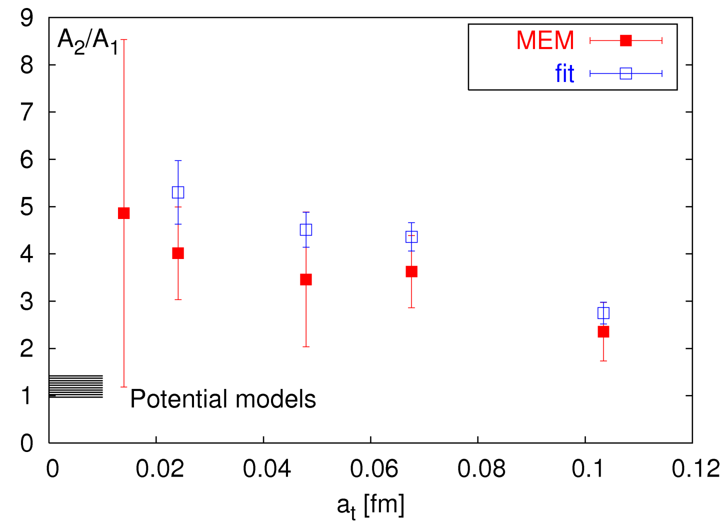
Pseudo-scalar:

$A_2/A_1 \simeq |R_{2S}(0)|^2/|R_{1S}(0)|^2$



Pseudo-scalar:

$A_2/A_1 \simeq |R'_{2P}(0)|^2/|R'_{1P}(0)|^2$



Using default model from the high energy part of the T=0 spectral functions :
resonances appears as small structures on top of the continuum,
almost no T-dependence in the PS spectral functions till $T \simeq 2.4T_c$